Hamiltonian constraint formulation of classical field theories

Václav Zatloukal

[1] FNSPE, Czech Technical University in Prague, Břehová 7, 115 19 Praha 1, Czech Republic
[2] Max Planck Institute for the History of Science, Boltzmannstr. 22, 14195 Berlin, Germany

[www.zatlovac.eu]

Variational principle

$(x^\mu, \phi^A) \equiv q \ldots$ partial observables

$q \in \mathcal{C} \ldots$ configuration space of dim. $N + D$

$\gamma \ldots$ D-dim. surfaces in $\mathcal{C}$ (cf. field configurations $\phi^A(x^\mu)$)

$P \ldots$ momentum multivector of grade D (mechanics: $D = 1$)

Geometric algebra [1, 2]**:** $ab = a \cdot b + a \wedge b$
$\cdot\ ,\ \wedge \ldots$ inner, outer product of multivectors
$\partial_q \ldots$ vector derivative (cf. gradient ∇)

Action functional:

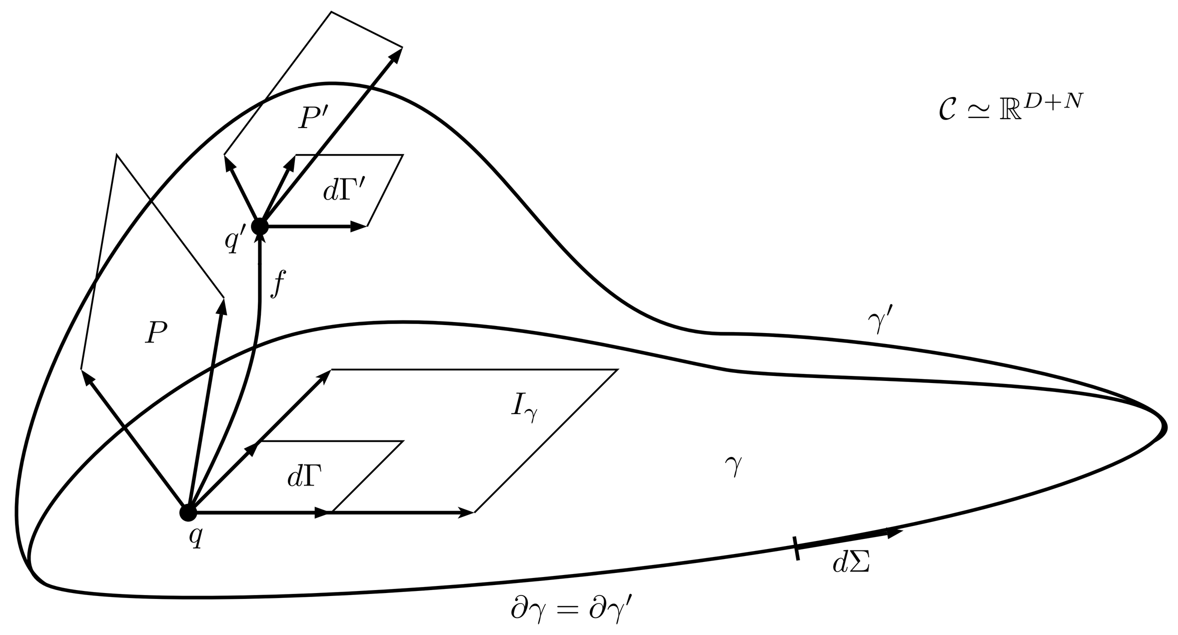
$$\mathcal{A}[\gamma, P] = \int_\gamma P(q) \cdot d\Gamma(q)$$

Hamiltonian constraint:

$$H(q, P(q)) = 0 \qquad \forall q \in \gamma$$

$$\Downarrow$$

Extremals are classical motions γ_{cl} [3]



Canonical equations of motion

Extended action $\mathcal{A}[\gamma, P, \lambda] = \int_\gamma [P(q) \cdot d\Gamma(q) - \lambda(q) H(q, P(q))] \quad \Rightarrow \quad$ extremize:

"Velocity ~ Momentum"

$$\lambda\, \partial_P H(q, P) = d\Gamma,$$

$$(-1)^D \lambda\, \dot{\partial}_q H(\dot{q}, P) = \begin{cases} d\Gamma \cdot \partial_q P & \text{for } D = 1 \\ (d\Gamma \cdot \partial_q) \cdot P & \text{for } D > 1, \end{cases}$$

$$H(q, P) = 0.$$

"Force = Change in momentum"

Local Hamilton-Jacobi theory

Solve

$$H(q, \partial_q \wedge S) = 0$$

for $D - 1$-vector-valued $S(q)$, then

$$\lambda\, \partial_P H(q, \partial_q \wedge S) = d\Gamma$$

defines a distribution of tangent planes of classical motions.

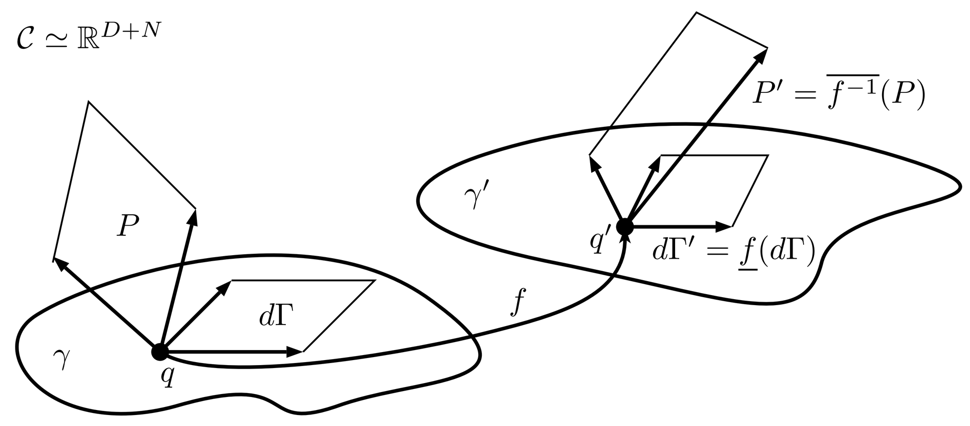
Solution $S(q; \alpha)$ depending on a continuous parameter $\quad \Rightarrow \quad \partial_\alpha S$ conserved.

Symmetries and conservation laws

Transformation $f : \mathcal{C} \to \mathcal{C}$

Differential: $\underline{f}(a; q) \equiv a \cdot \partial_q f(q)$

Adjoint: $\overline{f}(b; q) \equiv \partial_q f(q) \cdot b$



$(\gamma, P) \to (\gamma', P')$ preserves the action (since $d\Gamma' = \underline{f}(d\Gamma)$ and $P' := \overline{f^{-1}}(P)$)

$\Rightarrow$ If γ_{cl} classical motion of H, then γ'_{cl} classical motion of H': $H'(q', P') = H(q, P)$.

f is a symmetry if $H' = H$, i.e., $H(f(q), \overline{f^{-1}}(P; q)) = H(q, P)$.
For infinitesimal $f(q) = q + \varepsilon v(q)$

$$v \cdot \dot{\partial}_q H(\dot{q}, P) - \big(\dot{\partial}_q \wedge (\dot{v} \cdot P)\big) \cdot \partial_P H(q, P) = 0$$

Insertion of the canonical equations of motion yields
Conservation law:

$$d\Gamma \cdot \partial_q (P \cdot v) = 0 \quad \text{for } D = 1$$
$$(d\Gamma \cdot \partial_q) \cdot (P \cdot v) = 0 \quad \text{for } D > 1$$

Integral form:

$$P(q_2) \cdot v(q_2) - P(q_1) \cdot v(q_1) = 0 \quad \text{for } D = 1$$
$$\int_{\partial\gamma_{\text{cl}}} d\Sigma \cdot (P \cdot v) = 0 \quad \text{for } D > 1$$

$P \cdot v \ldots \sim$ Noether current

Example: Scalar field theory

$\mathcal{C} =$ spacetime $\oplus$ field space ($q = x + y$)
$I_x \ldots$ spacetime pseudoscalar, $\{e_a\}_{a=1}^N \ldots$ field space orthonormal basis

$$H(q, P) = P \cdot I_x + H_{\text{DW}}(q, P)$$

$H_{\text{DW}} \ldots$ De Donder-Weyl Hamiltonian: $I_x \cdot \partial_P H_{\text{DW}} = 0$, $(e_b \wedge e_a) \cdot \partial_P H_{\text{DW}} = 0$ $(\forall a, b)$

Motions and momenta as functions on the spacetime:

$$\gamma = \{q = x + y(x) \,|\, x \in \Omega\} \quad , \quad \mathsf{P}(x) \equiv P(x + y(x))$$

Canonical equations $\Rightarrow$ De Donder-Weyl equations

$$\partial_x y = I_x^{-1} \partial_P H_{\text{DW}} \quad , \quad (e_a I_x \partial_x) \cdot \mathsf{P} = (-1)^D e_a \cdot \partial_y H_{\text{DW}}$$

Scalar field Hamiltonian:

$$H_{\text{SF}}(q, P) = P \cdot I_x + \frac{1}{2} \sum_{a=1}^N \big(I_x \cdot (P \cdot e_a)\big)^2 + V(y)$$

Conservation law $\Rightarrow$ Continuity equation

$$\partial_x \cdot j(x) = 0$$

with the Noether current

$$j(x) \equiv -I_x \cdot \Big[\mathsf{P} \cdot \mathsf{v} + \dot{\partial}_x \wedge \big(\dot{y} \cdot (\mathsf{P} \cdot \mathsf{v})\big)\Big] \quad , \quad \mathsf{v}(x) \equiv v(x + y(x))$$

Example: String theory

$\mathcal{C} \ldots$ target space, $\gamma \ldots$ world-sheet

$$H_{\text{Str}}(P) = \frac{1}{2}(|P|^2 - \Lambda^2) \quad , \quad \text{where} \quad |P|^2 \equiv \widetilde{P} \cdot P$$

Reversion: $(a \ldots b)^\sim = b \ldots a$

Elimination of P and λ yields the Nambu-Goto action:

$$\mathcal{A}_{\text{Str}} = \pm \Lambda \int_\gamma |d\Gamma|$$

and the equation of motion:

$$(I_\gamma \cdot \partial_q) \cdot I_\gamma = 0 \quad , \quad \text{where} \quad I_\gamma \equiv d\Gamma / |d\Gamma|$$

References

[1] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, Springer (1987).
[2] C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, Cambridge Univ. Press (2007).
[3] C. Rovelli, *Quantum Gravity*, Cambridge Univ. Press (2004).
[4] V. Zatloukal, *Hamiltonian constraint formulation of classical field theories*, Adv. Applied Clifford Algebras, DOI: 10.1007/s00006-016-0663-0 (2016) [arXiv:1602.00468].

This work has been supported by the GAČR Grant No. GA14-07983S, the CTU in Prague Grant No. SGS13/217/OHK4/3T/14, and the DFG Grant No. KL 256/54-1.