Lecture notes for the course 02KTPA1

(Quantum field theory 1)

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Preface

These lecture notes serve as a reference for the course Quantum field theory 1 (02KTPA1) taught at the Czech Technical University in Prague (Faculty of Nuclear Sciences and Physical Engineering). They cover relativistic quantum mechanics (Part I) and basics of quantum field theory (Part II). This course takes 13 weeks in a 4+2 weekly scheme, meaning 4×50 minutes of lectures plus 2×50 minutes of exercises. Correspondingly, the lecture notes are divided into 13 chapters, with (solved) exercises provided at the end of each chapter.

I will be very grateful for reported errors or suggestions for improvement.

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Chapter 0

Overview of the course

0.1 Motivation and outline

Quantum field theory (QFT) is the latest step in the process of establishing fields, that is, quantities defined throughout spacetime, as the fundamental entities that constitute physical reality. Let us summarize this process.

1. It starts in **Newtonian gravity**, where one visualises gravitational field of a certain massive source as a collection of vectors, which force other matter to accelerate.

2. **In electrodynamics**, the electromagnetic field can exist even without sources, propagating as a wave with universal velocity c. If charged particles are present, there is no direct non-local action at a distance between them, but rather their interactions are mediated by the field. (Particle A locally disturbs the field, the signal propagates for a finite amount of time, and eventually hits particle B.)

3. The tension between Lorentz invariance of electromagnetic field equations and Galilean invariance of the equations of Newtonian mechanics results in the invention of **special relativity**. It asserts that the Lorentz transformations of the field theory should also apply to matter. This indicates that there could be wave-like aspects of material particles.

4. **Quantum mechanics** provides a precise mathematical model of particles as blurred 'wave-packets of probability', or *quanta*. However, it is inconsistent with special relativity, treating time as an ordinary real parameter t while promoting spatial coordinates to operators $\hat{X}_i$.

5. Meanwhile it is realized that the classical electromagnetic field paradigm is not consistent with black-body radiation experiments, and photoelectric effect. The conclusion is that the energy of electromagnetic waves does not change smoothly, but rather jumps in steps $\hbar\omega$ — the field is quantized, it is a collection of *photons*. (**Quantized electromagnetic field** then also explains other phenomena like spontaneous emission of radiation from atoms.)

Quantum field theory puts forward a model, where all spacetime coordinates $\boldsymbol{x}$ and t are real parameters, and the operator nature of quantum theory is carried by the field amplitudes $\hat{\phi}(\boldsymbol{x}, t)$. This model applies to both matter and radiation, thereby closing the conceptual gap between the two. Everything is fundamentally made of fields, although the field excitations, being quantized, posses certain particle-like characteristics. As a bonus, it is easy to describe processes where numbers and types of particles change, such as electron–positron annihilation, $e^- + e^+ \to 2\gamma$, or β-decay, $n \to p + e^- + \bar{\nu}_e$.

The unifying character of quantum field theory sounds attractive, yet by itself would not make for a course within standard physics curriculum. The actual *reason* for endorsing quantum field theory as the right description of Nature on its fundamental level is its ability to make testable predictions that are being confirmed by countless particle physics experiments.

But it's not only the Standard model of elementary particles and fundamental interactions between them that makes use of the idea of quantized fields. QFT methods are of great value also in condensed matter physics, which studies properties of solids and liquids, i.e., systems of macroscopically many particles. The two disciplines have cross-fertilized each other with ideas from condensed matter physics (e.g., renormalization, the Higgs mechanism) transferred to particle physics, and vice versa (e.g., the concept of quasiparticles). The common QFT language indeed substantially facilitates this communication.

The aim of this course is to provide a concise introduction into quantum field theory, keeping in mind that there will be a follow-up advanced course (02KTPA2) that will present more advanced techniques, and more modern topics (some of which will be briefly outlined — e.g., renormalization, Yang-Mills theories, field-theoretic Feynman path integral).

The field (or wave) equations that we will be dealing with can (and will be) used in different physical contexts. For example, the Klein-Gordon equation $(\Box + m^2)\phi = 0$ can be used to describe a spinless relativistic quantum particle, or a classical field (an oscillating membrane), or a scalar quantum field. To minimize the possible confusion, these notes are strictly partitioned. Part I (Chapters 1–4) explores single-particle relativistic quantum mechanics. Its main purpose is to get familiar with the Dirac equation and Dirac spinors. Part II starts with Chapter 5 on classical field theory. All quantities there are classical, no operators.

Actual quantum field theory is a subject of Chapters 6–13.2. To introduce the somewhat abstract concept of quantum fields we first analyse a generic discrete system of coupled harmonic oscillators, as it only takes elementary quantum mechanics to quantize their position degrees of freedom, $q_i \to \hat{q}_i$. When the oscillators are arranged in a linear chain, the continuum limit results in a string, whose classical configurations are described by a field $\phi(x)$, and upon quantization we obtain a quantum field $\hat{\phi}(x)$. Although in practice one usually starts directly from a continuum theory with certain Lagrangian density, the picture of quantum field as a (potentially continuous) collection of coupled oscillators provides a valuable visual intuition.

This picture is in fact perfectly fitting the quantized motion of atoms that form a crystal. These atoms vibrate around their equilibrium lattice positions, and their collective excitations (called *phonons*) are quantized in a similar way as the photons of electromagnetic field. Whether spacetime itself has fundamentally a discrete lattice structure, i.e., forms a kind of 'world-crystal' whose vibrations give rise to elementary particles, is an open question.

0.2 Selected preliminaries from previous courses

Quantum field theory combines classical field theory, special relativity, and quantum mechanics, and uses relatively advanced mathematical techniques. To narrow down a little let us point out some concrete elements from these subjects that will be particularly useful (in order of appearance during the course):

1. The formalism of special relativity (TEF2)
2. Pauli matrices and spin (KVAN1)

3. Lagrangian and Hamiltonian formulation of classical mechanics (TEF1)
4. Action principle in classical field theory (TEF2)
5. Small vibrations and normal coordinates (VOAF)
6. Quantum harmonic oscillator, ladder operators (KVAN1)
7. Dirac delta function and Green's functions of differential operators (RMF)
8. Heisenberg and Dirac picture of quantum mechanics (KVAN2)

0.3 Literature

This course (02KTPA1) was previously taught by doc. Petr Jizba whose lecture notes [1] contain additional material on certain topics, and so can serve as a complementary reading.

There are many monographs on quantum field theory. These lecture notes mainly follow the approach (as well as notational conventions) of Greiner and Reinhardt [2]. Other standard sources to mention are Itzykson and Zuber [3], Peskin and Schroeder [4], and Weinberg [5]. A very insightful book with short but very engaging chapters covering a great variety of topics is from Zee [6]. For a more pedagogical introduction the lecture notes by Hořejší [7] or Tong [8] are recommendable.

In any case, the reader should be warned that there are many conventional choices to make when building quantum field theory, and the resulting set of conventions will typically differ from one source to another.

0.4 Conventions

Let us summarize some conventions employed in these lecture notes. More will be said in Section 1.1 on special relativity.

1. With exception of Chapters 1 and 6 we will work in the so-called 'natural' physical units, i.e., set $\hbar = c = 1$. Thus, for example, $E = m(c^2)$, $E = (\hbar)\omega$, $x^0 = (c)t$, etc. (Length has the same unit as time, and this in turn is inverse to the unit of mass and energy.)
2. We denote by $\delta(.)$ the Dirac delta function in any dimension, its dimensionality being indicated by the argument, e.g., $\delta(\boldsymbol{x} - \boldsymbol{y}) = \delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^3 - y^3)$.
3. Einstein summation convention by default applies to various types of indices that we will encounter: spatial indices $i, j, \ldots$, spacetime indices $\mu, \nu, \ldots$, internal field indices $r, s, \ldots$, spinor indices $\alpha, \beta, \ldots$, as well as to indices $a, b, \ldots$ enumerating the symmetry generators (Lie algebra matrices).

Part I

Relativistic Quantum Mechanics

Chapter 1

Relativistic wave equations

1.1 Relativistic notation

By default, we will work with flat relativistic spacetime, i.e., a set of points (or rather coordinates thereof)

$$x = (x^\mu) = (x^0, x^1, x^2, x^3) = (ct, \boldsymbol{x}), \tag{1.1}$$

where $\boldsymbol{x} = (x^i)$ is the spatial part. It is customary to drop the parenthesis and write, e.g., x^μ instead of (x^μ) for brevity. Greek (spacetime) indices $\mu, \nu, \rho, \sigma, \ldots$ run from 0 to 3, and Latin (spatial) indices $i, j, k, \ell, \ldots$ from 1 to 3.

We will use the 'mostly-negative' convention for Minkowski metric,

$$\mathsf{g} = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \mathrm{diag}(1, -1, -1, -1), \tag{1.2}$$

common in particle physics. We denote by $(g^{\mu\nu}) = \mathsf{g}^{-1}$ the inverse metric, i.e., $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$, where the Kronecker symbol δ^μ_ρ represents the identity matrix. In our case of flat spacetime $g^{\mu\nu} = g_{\mu\nu}$. Application of the metric translates into raising or lowering of indices, e.g.,

$$x_\mu = g_{\mu\nu} x^\nu = (ct, -\boldsymbol{x}). \tag{1.3}$$

Four-momentum and the respective quantum operator read

$$p^\mu = \left(\frac{E}{c}, \boldsymbol{p} \right) \quad , \quad \hat{p}_\mu = i\hbar \frac{\partial}{\partial x^\mu} = \left(\frac{i\hbar}{c} \frac{\partial}{\partial t}, i\hbar \nabla \right). \tag{1.4}$$

Note that the operator ∇ is defined with lower index, $\nabla = (\partial_1, \partial_2, \partial_3)$, where $\partial_i \equiv \frac{\partial}{\partial x^i}$, and $\hat{\boldsymbol{p}} = (\hat{p}^i) = -i\hbar\nabla$ is the spatial momentum operator of quantum mechanics. We also adopt the standard notation

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \quad , \quad \Box \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta = \partial^\mu \partial_\mu \tag{1.5}$$

for the Laplace, and d'Alambert operator, respectively.

The Minkowski inner product of four-vectors a and b can be expressed in any of the following equivalent ways:

$$\begin{aligned} a \cdot b \equiv g_{\mu\nu} a^\mu b^\nu = g^{\mu\nu} a_\mu b_\nu = a^\mu b_\mu = a_\mu b^\mu = a_0 b^0 + a_1 b^1 + a_2 b^2 + a_3 b^3 \\ = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a_0 b_0 - a_i b_i. \end{aligned} \tag{1.6}$$

The Minkowski square is written as $a^2 \equiv a^\mu a_\mu$.

Lorentz transformations

$$x'^\mu = L^\mu{}_\nu x^\nu, \quad \text{or, in short,} \quad x' = \mathsf{L} x, \tag{1.7}$$

are linear transformations that preserve the Minkowski inner product:

$$a' \cdot b' = a \cdot b \quad \rightarrow \quad g_{\mu\nu} L^\mu{}_\rho L^\nu{}_\sigma = g_{\rho\sigma}. \tag{1.8}$$

Simple matrix 'gymnastics' yields

$$L_\nu{}^\rho L^\nu{}_\sigma = \delta^\rho_\sigma \quad \rightarrow \quad (\mathsf{L}^{-1})^\rho{}_\nu = L_\nu{}^\rho. \tag{1.9}$$

Taking L^{-1} instead of L in (1.8), and raising/lowering all indices we obtain

$$g_{\mu\nu} (\mathsf{L}^{-1})^\mu{}_\rho (\mathsf{L}^{-1})^\nu{}_\sigma = g_{\rho\sigma} \quad \rightarrow \quad g^{\mu\nu} L^\rho{}_\mu L^\sigma{}_\nu = g^{\rho\sigma}. \tag{1.10}$$

Standard Pauli matrices will have the index upstairs,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.11}$$

and we will often use the notation $\boldsymbol{n} \cdot \boldsymbol{\sigma} \equiv n^i \sigma^i$. Pauli matrices satisfy the identity

$$\sigma^i \sigma^j = \frac{1}{2} \{\sigma^i, \sigma^j\} + \frac{1}{2} [\sigma^i, \sigma^j] = \delta_{ij} + i\, \varepsilon_{ijk} \sigma^k, \tag{1.12}$$

where $[A, B] \equiv AB - BA$ is the commutator, and $\{A, B\} \equiv AB + BA$ the anti-commutator of two matrices (or operators). We will always write the 'three-dimensional' Levi-Civita permutation symbol ε_{ijk} (with $\varepsilon_{123} = 1$) with indices downstairs (same for the 'three-dimensional' Kronecker δ_{ij}), although the vertical position of indices on the two sides of an equation will not always match. We also define the totally anti-symmetric 'four-dimensional' Levi-Civita symbol $\varepsilon^{\mu\nu\rho\sigma}$ (with $\varepsilon^{0123} = 1$), whose indices may be raised or lowered with the Minkowski metric.

1.2 Klein-Gordon equation

In quantum mechanics the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m} \Delta + V(\boldsymbol{x}) \right) \psi(\boldsymbol{x}, t) \tag{1.13}$$

corresponds to quantisation of the non-relativistic energy relation

$$E = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x}), \tag{1.14}$$

where energy and momentum are turned into quantum operators

$$\hat{E} = i\hbar\frac{\partial}{\partial t} \quad , \quad \hat{\boldsymbol{p}} = -i\hbar\nabla. \tag{1.15}$$

In order to obtain a relativistic wave equation we start by considering free particles with the relativistic energy-momentum relation

$$p^\mu p_\mu = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2c^2, \tag{1.16}$$

where m is the particle's rest mass, and promote it to a dispersion relation of the sought-for wave equation. (The term 'dispersion relation' is known from wave mechanics as the dependence between frequency ω and wave-vector's magnitude $|\boldsymbol{k}|$. In quantum mechanics, the energy of a plane wave is related to the frequency by $E = \hbar\omega$, and the momentum to the wave-vector by $\boldsymbol{p} = \hbar\boldsymbol{k}$.) The group velocity of wave-packets (or 'quantum particles') is now given by

$$v_g = \frac{\partial\omega}{\partial|\boldsymbol{k}|} = \frac{\partial E}{\partial|\boldsymbol{p}|} = \frac{|\boldsymbol{p}|}{E}c^2, \tag{1.17}$$

which coincides with the velocity of relativistic particles, given by

$$\boldsymbol{v} = \frac{\boldsymbol{p}}{\gamma m} = \frac{\boldsymbol{p}}{E}c^2. \tag{1.18}$$

Note that in non-relativistic quantum mechanics we have the free-particle dispersion relation $E = \frac{\boldsymbol{p}^2}{2m}$, which implies $v_g = \frac{|\boldsymbol{p}|}{m}$.

Inspired by the quantization rules of non-relativistic quantum mechanics we replace the four-momentum p_μ by an operator $\hat{p}_\mu = i\hbar\partial_\mu$, and arrive at the *Klein-Gordon equation* for a free (spinless) relativistic particle,

$$\hat{p}^\mu\hat{p}_\mu\psi = m^2c^2\psi \quad \to \quad \left(\partial^\mu\partial_\mu + \frac{m^2c^2}{\hbar^2}\right)\psi(x) = 0, \tag{1.19}$$

described by a complex-valued wave-function $\psi(x)$. Here, the quantity $\frac{\hbar}{mc}$ is the (reduced) Compton wavelength of the particle. It can be shown (see Exercise 2) that in the non-relativistic limit of dominant rest energy, $E - mc^2 \ll mc^2$, the Klein-Gordon equation reduces to the Schrödinger equation for a free (spinless) particle.

In fact, the relativistic equation (1.19) was first considered by Schrödinger, who, however, got discouraged after calculating energy levels of the hydrogen atom, which did not match experimental results on the fine structure level. (Fine structure receives corrections due to relativity as well as spin of the electron, and the latter is not captured by the 'spinless' relativistic equation (1.19).) Schrödinger therefore decided to put forward his non-relativistic equation, which was able to explain the gross structure of the hydrogen energy spectrum, and the relativistic equation was published soon after by Klein and Gordon (more about the history in [5, Ch. 1.1].)

It is worth to remark that were it not for the 'mass term' $m^2c^2/\hbar^2$ Eq. (1.19) would have the form of an ordinary wave equation. With the mass term present it formally corresponds to a wave equation for classical waves propagating in plasma [9, Ch. 6.4].

The Klein-Gordon equation is a linear differential equation with constant coefficients, and as such admits solutions in the form of plane waves

$$\psi_p(x) = A_p \exp\left(-\frac{i}{\hbar}\,p\cdot x\right), \quad \text{where} \quad p^\mu p_\mu = m^2c^2, \quad \text{and} \quad \hat{p}_\mu\psi_p = p_\mu\psi_p. \tag{1.20}$$

The dispersion relation can be cast as

$$E = \pm c\sqrt{m^2c^2 + \boldsymbol{p}^2}, \tag{1.21}$$

hence there exist solutions with both positive and negative values of energy, the negative ones being related to the existence of antiparticles (as we shall see in Sec. 7.3). (Note that, unlike in classical physics, in quantum physics the negative-energy part cannot be simply ignored since it is possible for a quantum system to cross the finite energy gap from positive to negative energies, much like an electron can jump between the energy levels of an atom.)

Next we look for a conserved current j_μ corresponding to the Klein-Gordon equation. We multiply Eq. (1.19) on the left by ψ^*, and subtract a complex-conjugate equation,

$$\psi^* \left(\partial^\mu\partial_\mu + \frac{m^2c^2}{\hbar^2}\right)\psi - \psi\left(\partial^\mu\partial_\mu + \frac{m^2c^2}{\hbar^2}\right)\psi^* = 0. \tag{1.22}$$

The mass terms cancel, and the rest can be cast as

$$\partial^\mu j_\mu = 0, \quad \text{where} \quad j_\mu \equiv \frac{i\hbar}{2m}(\psi^*\partial_\mu\psi - \psi\partial_\mu\psi^*), \tag{1.23}$$

and where the constant factor $i\hbar/2m$ has been added to match, in the non-relativistic limit, the probability current of the Schrödinger equation (see Exercise 2). Eq. (1.23) has the form of a continuity equation,

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\boldsymbol{j} = 0, \quad \text{where} \quad \rho \equiv \frac{j_0}{c} = \frac{i\hbar}{2mc^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right). \tag{1.24}$$

It would be a natural guess to interpret ρ as the probability density. However, it is not hard to realise that ρ can in fact be negative. Indeed, plugging, for example, the plane-wave solution ψ_p we find

$$\rho_p = |A_p|^2\frac{E}{mc^2}, \tag{1.25}$$

where E can be both positive or negative. A better interpretation is provided by multiplying ρ by charge e, and regarding $e\rho$ as the charge density, which is allowed to be positive, negative or zero.

1.3 First-order equations

The fact that the Klein-Gordon equation is of second order in time derivative means that it does not determine the time evolution of a given initial configuration $\psi(\boldsymbol{x}, t_0)$ unless we provide also $\partial_t\psi(\boldsymbol{x}, t_0)$. This contrasts with the Schrödinger equation, which determines time evolution of any given initial state. Therefore we look for a relativistic wave equation that would feature only first-order time derivatives.

There are several ways in which one could attempt to resolve this issue. We list three alternatives, where the last one leads to the celebrated Dirac equation, which is of great physical importance.

1.3.1 Direct square root

One obvious way to proceed is to rewrite the classical relativistic energy-momentum relation $p^\mu p_\mu = m^2c^2$ in the form

$$H \equiv E = \sqrt{m^2c^4 + \boldsymbol{p}^2c^2}, \tag{1.26}$$

and consider the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi(\boldsymbol{x}, t) \quad \text{with Hamiltonian} \quad \hat{H} = \sqrt{m^2c^4 - \hbar^2c^2\Delta}. \tag{1.27}$$

However, this equation treats space and time on different footing (this asymmetry is not what we would expect of a relativistic theory). To make sense of the right-hand side, the square root of the differential operator has to be interpreted as an infinite Taylor series (recall the formula $(1+x)^\alpha = 1 + \alpha x + \alpha(\alpha-1)x^2 + \ldots$) of spatial derivatives acting on $\psi(\boldsymbol{x}, t)$.

1.3.2 Two coupled equations

The order of any differential equation can be lowered by introducing additional unknown functions. In particular, the Klein-Gordon equation is equivalent to a Schrödinger-type equation (see Exercise 3)

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi, \quad \text{where} \quad \hat{H} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\frac{\hat{\boldsymbol{p}}^2}{2m} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}mc^2, \tag{1.28}$$

and where Ψ is a two-component wave-function.

1.3.3 Dirac's method

As we know from quantum mechanics, the square of any spatial vector can be represented as (recall the identity (1.12))

$$(\sigma^i p_i)(\sigma^j p_j) = \delta_{ij}p_ip_j = p_ip_i, \tag{1.29}$$

where the identity matrix is implicit in the expressions. In 1928, Paul Dirac realised that, similarly, in spacetime, for any four-vector it holds that

$$(\gamma^\mu p_\mu)(\gamma^\nu p_\nu) = \gamma^\mu\gamma^\nu p_\mu p_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}p_\mu p_\nu = g^{\mu\nu}p_\mu p_\nu = p^\mu p_\mu, \tag{1.30}$$

provided that the objects γ^μ satisfy the relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad (\forall \mu, \nu). \tag{1.31}$$

With p_μ replaced by the operators $\hat{p}_\mu = i\hbar\partial_\mu$ this allowed him to factorise (or, take a square-root of) the Klein-Gordon differential operator,

$$\hat{p}^\mu\hat{p}_\mu - m^2c^2 = (\gamma^\mu\hat{p}_\mu + mc)(\gamma^\nu\hat{p}_\nu - mc), \tag{1.32}$$

and postulate the relativistic wave equation

$$(i\hbar\gamma^\mu\partial_\mu - mc)\Psi(x) = 0, \tag{1.33}$$

which implies the Klein-Gordon equation for $\Psi(x)$.

What are the γ^μ's? They can't be just ordinary commuting numbers since then we couldn't have $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0$ for $\mu \neq \nu$. It turns out that the relation (1.31) is satisfied by certain 4 by 4 matrices (the Dirac matrices), and hence the (Dirac) wave-function Ψ has four components.

The Dirac equation is arguably one of the most important equations of the 20th century physics. We shall study it thoroughly in Chapters 3 and 4, where we will see that it describes spin-$\frac{1}{2}$ particles, and correctly predicts the fine structure spectrum of the hydrogen atom.

1.4 Exercises

Exercise 1. *Lorentz invariance of Klein-Gordon equation.* Show that the Klein-Gordon equation (with $\hbar = c = 1$)

$$(\partial^\mu \partial_\mu + m^2)\psi(x) = 0 \tag{1.34}$$

is invariant under Lorentz transformations $x'^\mu = L^\mu{}_\nu x^\nu$.

Solution:

We assume that the wave-function ψ transforms as a scalar field under Lorentz transformations:

$$\psi'(x') = \psi(x), \quad \text{hence} \quad \frac{\partial \psi'(x')}{\partial x'^\mu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial \psi(x)}{\partial x^\rho} = (\mathsf{L}^{-1})^\rho{}_\mu \partial_\rho \psi(x). \tag{1.35}$$

Denoting $\partial'_\mu \equiv \frac{\partial}{\partial x'^\mu}$, we find

$$(g^{\mu\nu}\partial'_\mu\partial'_\nu + m^2)\psi'(x') = \Big(g^{\mu\nu}(\mathsf{L}^{-1})^\rho{}_\mu(\mathsf{L}^{-1})^\sigma{}_\nu\partial_\rho\partial_\sigma + m^2\Big)\psi(x) = (g^{\rho\sigma}\partial_\rho\partial_\sigma + m^2)\psi(x) = 0, \tag{1.36}$$

where we have used the defining property of the Lorentz transformations, Eq. (1.10).

Exercise 2. *Non-relativistic limit of Klein-Gordon equation.* Plug the ansatz

$$\psi(\boldsymbol{x}, t) = \varphi(\boldsymbol{x}, t)\exp\left(-\frac{i}{\hbar}mc^2 t\right) \tag{1.37}$$

into the Klein-Gordon equation (1.19), and show that it reduces to the Schrödinger equation in the regime when φ oscillates much more slowly in time than the exponential factor, i.e., $i\hbar\partial_t\varphi \ll mc^2\varphi$.

What does the Klein-Gordon current (1.23) reduce to in this approximation?

Solution:

We calculate

$$\begin{aligned}
\partial_t\psi(\boldsymbol{x}, t) &= e^{-\frac{i}{\hbar}mc^2 t}\Big(\partial_t - \frac{i}{\hbar}mc^2\Big)\varphi(\boldsymbol{x}, t), \\
\partial_t^2\psi(\boldsymbol{x}, t) &= e^{-\frac{i}{\hbar}mc^2 t}\Big(\partial_t^2 - 2\frac{i}{\hbar}mc^2\partial_t - \frac{m^2c^4}{\hbar^2}\Big)\varphi(\boldsymbol{x}, t).
\end{aligned} \tag{1.38}$$

Non-relativistic approximation consist in neglecting the ∂_t^2 term, since the time variation of φ is much slower than the rest energy contributions. The Klein-Gordon equation thus reads

$$\frac{1}{c^2}\partial_t^2\psi - \Delta\psi + \frac{m^2c^2}{\hbar^2}\psi \approx e^{-\frac{i}{\hbar}mc^2 t}\left[\Big(-2\frac{i}{\hbar}m\,\partial_t - \frac{m^2c^2}{\hbar^2}\Big)\varphi - \Delta\varphi + \frac{m^2c^2}{\hbar^2}\varphi\right] = 0, \tag{1.39}$$

which can be rearranged to display the free-particle Schrödinger equation

$$i\hbar\partial_t\varphi = -\frac{\hbar^2}{2m}\Delta\varphi. \tag{1.40}$$

Regarding the Klein-Gordon four-current, we have

$$\rho = \frac{i\hbar}{2mc^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) = \frac{i\hbar}{2mc^2}\left[\varphi^*\Big(\partial_t - \frac{i}{\hbar}mc^2\Big)\varphi - \varphi\Big(\partial_t + \frac{i}{\hbar}mc^2\Big)\varphi^*\right] \approx \varphi^*\varphi, \tag{1.41}$$

where we have neglected ∂_t against $\frac{i}{\hbar}mc^2$, and

$$j^i = \frac{i\hbar}{2m}\big(\psi^*\partial^i\psi - \psi\,\partial^i\psi^*\big) = \frac{i\hbar}{2m}\big(-\varphi^*\partial_i\varphi + \varphi\,\partial_i\varphi^*\big) \quad \to \quad \boldsymbol{j} = \frac{i\hbar}{2m}\big(\varphi\,\nabla\varphi^* - \varphi^*\nabla\varphi\big), \tag{1.42}$$

recovering the non-relativistic probability density and current, respectively.

Exercise 3. *Feshbach-Villars representation of Klein-Gordon equation.* Rewrite the Klein-Gordon equation in a Schrödinger-like form — as an equation of first order in ∂_t for a two-component wave-function.

Solution:

Let us put $\hbar = c = 1$. Introducing a function $\xi = \frac{i}{m}\partial_t\psi$, the Klein-Gordon equation $(\partial_t^2 - \Delta + m^2)\psi = 0$ can be rewritten as a pair of equations

$$\begin{aligned} i\partial_t\psi &= m\xi \\ i\partial_t\xi &= -\frac{1}{m}\Delta\psi + m\psi. \end{aligned} \tag{1.43}$$

Their addition and subtraction yield, respectively, (denoting $\varphi \equiv \psi + \xi$, and $\chi \equiv \psi - \xi$)

$$\begin{aligned} i\partial_t\varphi &= -\frac{1}{2m}\Delta(\varphi + \chi) + m\varphi \\ i\partial_t\chi &= \frac{1}{2m}\Delta(\varphi + \chi) - m\chi. \end{aligned} \tag{1.44}$$

In matrix form, and with $\hbar$ and c restored,

$$i\hbar\partial_t\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = -\frac{\hbar^2}{2m}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\Delta\begin{pmatrix}\varphi\\ \chi\end{pmatrix} + mc^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}. \tag{1.45}$$

Chapter 2

Relativistic transformations

Before discussing the Dirac equation, and exploring its consequences, we set up some mathematics for working in $3+1$-dimensional spacetime. These constructions will be motivated by analogies with three-dimensional space.

We set $\hbar = c = 1$ from now on.

2.1 Lie groups in physics

2.1.1 Rotations in 3D

Consider rotations in a three-dimensional space, i.e., elements of the group $SO(3)$. To start with, take the rotation axis to be the z axis:

$$\mathsf{R}_3(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{|\alpha|\ll 1}{\approx} \mathbb{I} - i\alpha\mathsf{M}_3, \quad \text{where} \quad \mathsf{M}_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

is the *infinitesimal generator* of rotations around the z axis. It seems redundant to introduce the imaginary unit i in front of the generator (after all, we are dealing with classical spatial rotations). Nonetheless, this convention is common in physics, and so we adopt it.

Similarly, we can infer rotation generators around all three axes. Their matrix elements, and commutation relations read

$$(\mathsf{M}_i)_{jk} = -i\varepsilon_{ijk} \quad , \quad [\mathsf{M}_i, \mathsf{M}_j] = i\varepsilon_{ijk}\mathsf{M}_k. \tag{2.2}$$

A finite rotation around an arbitrary unit vector $\boldsymbol{n}$ can be obtained as a succession of many rotations by small (eventually infinitesimal) angle:

$$\mathsf{R}_{\boldsymbol{n}}(\alpha) = \left(\mathsf{R}_{\boldsymbol{n}}(\tfrac{\alpha}{N})\right)^N = \lim_{N\to\infty}\left(\mathbb{I} - i\frac{\alpha}{N}n^j\mathsf{M}_j\right)^N = e^{-i\alpha n^j \mathsf{M}_j}. \tag{2.3}$$

As we know from quantum mechanics, there are many triples of matrices that satisfy the 'angular momentum' commutation relations in Eq. (2.2), and they are distinguished by the value of spin: $\frac{1}{2}, 1, \frac{3}{2}, \ldots$. Smallest such matrices are obtained easily from the Pauli matrices:

$$[\sigma^i, \sigma^j] = 2i\varepsilon_{ijk}\sigma^k \quad \rightarrow \quad \left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\varepsilon_{ijk}\frac{\sigma^k}{2}. \tag{2.4}$$

Now

$$\mathsf{U}_{\boldsymbol{n}}(\alpha) \equiv \exp\left(-\frac{i}{2}\alpha\, \boldsymbol{n}\cdot\boldsymbol{\sigma}\right) \in SU(2), \quad \text{where} \quad \boldsymbol{n}\cdot\boldsymbol{\sigma} \equiv n^j\sigma^j, \tag{2.5}$$

is an exponential analogous to $\mathsf{R}_{\boldsymbol{n}}(\alpha)$, where the generators M_j have been replaced by $\frac{1}{2}\sigma^j$.

Vectors v^i can be rotated using the $SU(2)$ matrices if we represent them by traceless Hermitian matrices (or 'Pauli vectors')

$$\mathsf{V} = v^i\sigma^i = \begin{pmatrix} v^3 & v^1 - iv^2 \\ v^1 + iv^2 & -v^3 \end{pmatrix}, \quad \text{with} \quad \det\mathsf{V} = -\boldsymbol{v}^2, \tag{2.6}$$

since then

$$\mathsf{U}^\dagger_{\boldsymbol{n}}(\alpha)\,\sigma^i\,\mathsf{U}_{\boldsymbol{n}}(\alpha) = \big(\mathsf{R}_{\boldsymbol{n}}(\alpha)\big)_{ij}\sigma^j \quad \to \quad \mathsf{U}_{\boldsymbol{n}}(\alpha)\,\mathsf{V}\,\mathsf{U}^\dagger_{\boldsymbol{n}}(\alpha) = \sigma^j\big(\mathsf{R}_{\boldsymbol{n}}(\alpha)\big)_{ji}v^i. \tag{2.7}$$

We will prove this in Exercise 4, and only note here that the transformation $\mathsf{V} \mapsto \mathsf{U}\mathsf{V}\mathsf{U}^\dagger$, for U unitary, maps Pauli vectors to Pauli vectors, and preserves the norm:

$$\mathrm{Tr}(\mathsf{U}\mathsf{V}\mathsf{U}^\dagger) = \mathrm{Tr}\,\mathsf{V} = 0 \quad , \quad (\mathsf{U}\mathsf{V}\mathsf{U}^\dagger)^\dagger = \mathsf{U}\mathsf{V}\mathsf{U}^\dagger \quad , \quad \det(\mathsf{U}\mathsf{V}\mathsf{U}^\dagger) = \det\mathsf{V}. \tag{2.8}$$

Moreover, U and $-\mathsf{U}$ yield the same rotation ($SU(2)$ is a double cover of $SO(3)$).

We have seen that rotations in a 3D (Euclidean) space can be represented either by 3 by 3 matrices $\in SO(3)$, or by 2 by 2 matrices $\in SU(2)$. The latter can, in addition, act on (Pauli) spinors, i.e., on elements of $\mathbb{C}^2$.

2.1.2 Lie groups and Lie algebras

The matrix groups $SO(3)$ and $SU(2)$ are instances of *Lie groups*, with their generators M_i and $\frac{1}{2}\sigma^i$ spanning the respective *Lie algebras* $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$. Having the same commutation relations, Eqs. (2.2) and (2.4), we identify $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ as two *representations* of the same (abstract) Lie algebra, usually denoted $\mathfrak{su}(2)$.

In general, a Lie group is a group whose elements depend analytically on a finite number of parameters λ_a. Near the identity, the group elements can be approximated by linear combinations of generators A_a — elements of the corresponding Lie algebra with commutator as the product operation. The generators satisfy certain commutation rules

$$[\mathsf{A}_a, \mathsf{A}_b] = f_{abc}\mathsf{A}_c, \tag{2.9}$$

where f_{abc} are referred to as the 'structure constants' as they determine the structure of the algebra. Finite transformations are then obtained via exponentiation of the generators (although, this operation may not always cover the entire Lie group).

In physics the Lie groups are typically groups of transformations that act on spacetime (rotations, Lorentz transformations), or on an 'internal space' of the theory. As an example, the set of N by N real antisymmetric matrices is closed under commutation, and so forms a Lie algebra. The corresponding Lie group is a group of rotations in $\mathbb{R}^N$ (this can correspond to a field with N components), since its elements e^{A} are orthogonal, and have determinant 1:

$$(e^{\mathsf{A}})^T = e^{\mathsf{A}^T} = e^{-\mathsf{A}} = (e^{\mathsf{A}})^{-1}, \quad \text{and} \quad \det e^{\mathsf{A}} = e^{\mathrm{Tr}\,\mathsf{A}} = e^0 = 1. \tag{2.10}$$

(Here, the relation $\det\exp = \exp\mathrm{Tr}$ can be easily proved by diagonalization $\mathsf{A} = \mathsf{P}\mathsf{D}\mathsf{P}^{-1}$ if A is diagonalizable, or with a help of the Jordan normal form for A generic.) In physics it is common to use Hermitian generators $\mathsf{T}_a = i\mathsf{A}_a$, so that

$$e^{\lambda_a\mathsf{A}_a} = e^{-i\lambda_a\mathsf{T}_a} \quad \text{and} \quad [\mathsf{T}_a, \mathsf{T}_b] = if_{abc}\mathsf{T}_c. \tag{2.11}$$

The fact that a group is closed under multiplication is reflected in its algebra being closed under commutation. The link is provided by the Baker-Campbell-Hausdorff formula for multiplication of matrix exponentials:

$$e^{\mathsf{A}}e^{\mathsf{B}} = \exp\Big(\mathsf{A}+\mathsf{B}+\frac{1}{2}[\mathsf{A},\mathsf{B}]+\frac{1}{12}[\mathsf{A}-\mathsf{B},[\mathsf{A},\mathsf{B}]]+\dots\Big), \tag{2.12}$$

where the dots represent higher commutators. This formula can be rather cumbersome, but we will actually only need a restricted version:

$$e^{\mathsf{A}}e^{\mathsf{B}} = e^{\mathsf{A}+\mathsf{B}}e^{\frac{1}{2}[\mathsf{A},\mathsf{B}]} \qquad \text{provided} \qquad [\mathsf{A},[\mathsf{A},\mathsf{B}]] = [\mathsf{B},[\mathsf{A},\mathsf{B}]] = 0, \tag{2.13}$$

which we derive in Exercise 5 with a help of another very useful formula, the Campbell identity

$$e^{\mathsf{A}}\,\mathsf{B}\,e^{-\mathsf{A}} = \mathsf{B}+[\mathsf{A},\mathsf{B}]+\frac{1}{2!}[\mathsf{A},[\mathsf{A},\mathsf{B}]]+\dots = \sum_{n=0}^{\infty}\frac{\mathsf{K}_n}{n!}, \quad \text{where} \quad \mathsf{K}_0=\mathsf{B}, \quad \mathsf{K}_{n+1}=[\mathsf{A},\mathsf{K}_n] \quad (\forall n). \tag{2.14}$$

With the definitions (common in mathematics)

$$\mathrm{Ad}_{e^{\mathsf{A}}}:\mathsf{B}\mapsto e^{\mathsf{A}}\,\mathsf{B}\,e^{-\mathsf{A}} \quad , \quad \mathrm{ad}_{\mathsf{A}}:\mathsf{B}\mapsto[\mathsf{A},\mathsf{B}] \qquad \text{we can write succinctly} \qquad \mathrm{Ad}_{e^{\mathsf{A}}}(\mathsf{B}) = e^{\mathrm{ad}_{\mathsf{A}}}(\mathsf{B}). \tag{2.15}$$

2.2 Lorentz group and Lorentz algebra

Lorentz transformations $x'^{\mu} = L^{\mu}{}_{\nu}x^{\nu}$ are defined by the constraint (recall Eq. (1.8))

$$g_{\mu\nu}L^{\mu}{}_{\rho}L^{\nu}{}_{\sigma} = g_{\rho\sigma}, \quad \text{or, in matrix form,} \quad \mathsf{L}^T\mathsf{g}\,\mathsf{L} = \mathsf{g}. \tag{2.16}$$

To find the generators we consider infinitesimal transformations $L^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - i\varepsilon\ell^{\mu}{}_{\nu}$, and expand the constraint to obtain

$$g_{\mu\nu}(\delta^{\mu}_{\rho} - i\varepsilon\ell^{\mu}{}_{\rho})(\delta^{\nu}_{\sigma} - i\varepsilon\ell^{\nu}{}_{\sigma}) = g_{\rho\sigma} \quad \to \quad \ell_{\rho\sigma} = -\ell_{\sigma\rho}. \tag{2.17}$$

On the infinitesimal level, a Lorentz transformation is given by an antisymmetric matrix $\ell_{\rho\sigma}$, which can be linearly combined out of 6 independent (basis) matrices. We can write the linear combination using a pair of, effectively antisymmetrized, spacetime indices:

$$\ell_{\rho\sigma} = \frac{1}{2}\omega_{\mu\nu}(\mathsf{M}^{\mu\nu})_{\rho\sigma}, \quad \text{where} \quad \omega_{\mu\nu} = -\omega_{\nu\mu}, \quad \text{and} \quad (\mathsf{M}^{\mu\nu})^{\rho}{}_{\sigma} = i(g^{\mu\rho}\delta^{\nu}_{\sigma} - g^{\nu\rho}\delta^{\mu}_{\sigma}), \tag{2.18}$$

so that the basis matrices satisfy $\mathsf{M}^{\mu\nu} = -\mathsf{M}^{\nu\mu}$, $(\mathsf{M}^{\mu\nu})_{\rho\sigma} = -(\mathsf{M}^{\mu\nu})_{\sigma\rho}$. (The factor $\frac{1}{2}$ is included to compensate for double counting in the summation over μ and ν.)

With this choice of generators the infinitesimal Lorentz transformations determined by parameters $\omega_{\mu\nu}$ are simply given by

$$\ell_{\rho\sigma} = \frac{i}{2}\omega_{\mu\nu}(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\nu}_{\rho}\delta^{\mu}_{\sigma}) = i\omega_{\rho\sigma} \quad \to \quad x'^{\mu} \approx (\delta^{\mu}_{\nu} - i\varepsilon\ell^{\mu}{}_{\nu})x^{\nu} = x^{\mu} + \varepsilon\,\omega^{\mu}{}_{\nu}x^{\nu}. \tag{2.19}$$

Finite Lorentz transformations are obtained by exponentiation as in Eq. (2.3),

$$\mathsf{L} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathsf{M}^{\mu\nu}\right). \tag{2.20}$$

At this point it is worth to note that Lorentz transformations, defined by Eq. (2.16), fall into four separate classes. Depending on whether $\det \mathsf{L}$ is $+1$ or -1 they are called *proper* or *improper*, and depending on whether ${L^0}_0$ is ≥ 1 or ≤ -1 they are called *orthochronous* or *non-orthochronous*. The transformations obtained by exponentiation of generators in Eq. (2.20) form the proper orthochronous class, as they are continuously connected to the identity transformation (by shrinking $\omega_{\mu\nu}$ to zero). All other transformations in the Lorentz group can be obtained from these by using two discrete transformations,

$$\text{parity} \quad \mathsf{L}_P = \mathrm{diag}(1,-1,-1,-1), \qquad \text{and} \qquad \text{time reversal} \quad \mathsf{L}_T = \mathrm{diag}(-1,1,1,1). \tag{2.21}$$

Let us draw our attention to the Lorentz algebra. In Exercise 6 we show that the Lorentz generators $\mathsf{M}^{\mu\nu}$ obey the commutation relations

$$[\mathsf{M}^{\mu\nu}, \mathsf{M}^{\rho\sigma}] = -i(g^{\mu\rho}\mathsf{M}^{\nu\sigma} - g^{\mu\sigma}\mathsf{M}^{\nu\rho} + g^{\nu\sigma}\mathsf{M}^{\mu\rho} - g^{\nu\rho}\mathsf{M}^{\mu\sigma}). \tag{2.22}$$

Alternatively, one can pass from $\mathsf{M}^{\mu\nu}$ to $3+3$ generators

$$\mathsf{J}_i \equiv \frac{1}{2}\varepsilon_{ijk}\mathsf{M}^{jk} \quad \text{and} \quad \mathsf{K}_i \equiv \mathsf{M}^{0i}, \tag{2.23}$$

in terms of which the commutation relations read

$$\begin{aligned} [\mathsf{J}_i, \mathsf{J}_j] &= i\varepsilon_{ijk}\mathsf{J}_k, \\ [\mathsf{J}_i, \mathsf{K}_j] &= i\varepsilon_{ijk}\mathsf{K}_k, \\ [\mathsf{K}_i, \mathsf{K}_j] &= -i\varepsilon_{ijk}\mathsf{J}_k. \end{aligned} \tag{2.24}$$

(The relation between $\mathsf{M}^{\mu\nu}$ and $(\mathsf{J}_i, \mathsf{K}_i)$ is the same as between the Faraday tensor $F^{\mu\nu}$ and the magnetic and electric field (B_i, E_i) — see Eq. (4.2).) The generators J_i correspond to spatial rotations around the axes x^i, while K_i correspond to boosts along x^i — from the definition (2.18) we obtain

$$({\mathsf{J}_i})^j{}_k = -i\varepsilon_{ijk}, \quad ({\mathsf{J}_i})^0{}_\mu = ({\mathsf{J}_i})^\mu{}_0 = 0, \qquad \text{and} \qquad ({\mathsf{K}_i})^j{}_k = 0, \quad ({\mathsf{K}_i})^0{}_\mu = ({\mathsf{K}_i})^\mu{}_0 = i\delta^i_\mu. \tag{2.25}$$

The exponential representation of finite Lorentz transformations can be cast in terms of the generators J_i, K_i as follows:

$$\mathsf{L}(\omega_{\mu\nu}) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathsf{M}^{\mu\nu}\right) = \exp\left(-\frac{i}{2}\omega_{ij}\mathsf{M}^{ij} - i\,\omega_{0i}\mathsf{M}^{0i}\right) = \exp\left(-i\theta^i\mathsf{J}_i - i\zeta^i\mathsf{K}_i\right) = \mathsf{L}(\boldsymbol{\theta}, \boldsymbol{\zeta}) \tag{2.26}$$

where we have introduced the rotation parameters $\boldsymbol{\theta} = (\theta^1, \theta^2, \theta^3)$, and the boost parameters $\boldsymbol{\zeta} = (\zeta^1, \zeta^2, \zeta^3)$ via the relations

$$\omega_{ij} = \varepsilon_{ijk}\theta^k \quad , \quad \omega_{0i} = \zeta^i. \tag{2.27}$$

A rotation by angle α around a unit vector $\boldsymbol{n}$ is implemented by $\boldsymbol{\theta} = \alpha\boldsymbol{n}$, and a boost by velocity $\boldsymbol{v}$ is given by the rapidity (vector) $\boldsymbol{\zeta} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}\,\mathrm{arctanh}\,\frac{|\boldsymbol{v}|}{c}$ (see Exercise 7 for an example).

In Section 2.1.1 we saw that the 2 by 2 matrices $\frac{1}{2}\sigma^i$, relevant for the description of spin in quantum mechanics, satisfy the same commutation rules as the rotation generators M_i. Can we similarly find a simple representation of the commutation relations (2.24) for the Lorentz generators J_i, K_i? Clearly, the first line is satisfied again by $\frac{1}{2}\sigma^i$, and it is not hard to realize that for the boost generators we can take $\pm\frac{i}{2}\sigma^i$. Hence, we find two possibilities: the 'left-handed' representation

$$\mathsf{J}^L_i = \frac{\sigma^i}{2} \quad , \quad \mathsf{K}^L_i = -i\frac{\sigma^i}{2}, \tag{2.28}$$

and the 'right-handed' representation

$$\mathsf{J}_i^R = \frac{\sigma^i}{2} \quad , \quad \mathsf{K}_i^R = i\frac{\sigma^i}{2}. \tag{2.29}$$

In the Dirac theory one combines the two representations into 4 by 4 matrices

$$\mathsf{J}_i^{L\oplus R} = \begin{pmatrix} \frac{1}{2}\sigma^i & \mathbb{O} \\ \mathbb{O} & \frac{1}{2}\sigma^i \end{pmatrix} \quad , \quad \mathsf{K}_i^{L\oplus R} = \begin{pmatrix} -\frac{i}{2}\sigma^i & \mathbb{O} \\ \mathbb{O} & \frac{i}{2}\sigma^i \end{pmatrix}, \tag{2.30}$$

which again satisfy the commutation rules (2.24).

2.3 Clifford algebra

Clifford algebras found their way to physics through the work of Pauli on non-relativistic spinning particles, and through Dirac's relativistic theory, but they had been know in mathematics for about half a century.

Let us recall that in Pauli's theory the σ-matrices can be used to express the scalar product between any two three-vectors as

$$\boldsymbol{a}\cdot\boldsymbol{b} = \frac{1}{2}\{a^i\sigma^i, b^j\sigma^j\}, \quad \text{since} \quad \{\sigma^i,\sigma^j\} = 2\delta_{ij}. \tag{2.31}$$

Geometrically, the matrices σ^i represent an orthonormal basis in the 3D space, and their products generate the Clifford algebra of this space.

In spacetime we look for matrices γ^μ such that

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}, \quad \text{or} \quad \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2g^{\mu\nu}. \tag{2.32}$$

Finding them will allow us to write $a\cdot b = \frac{1}{2}\{\gamma^\mu a_\mu, \gamma^\nu b_\nu\}$, and hence implement the Dirac's idea of factorizing the Klein-Gordon operator. We shall see explicit representations of the γ-matrices in Section 2.3.1, but for the moment let us stay on an abstract level.

Taking products of γ^μ's, we obtain a set of matrices

$$\{\mathbb{I}\ ,\ \gamma^\mu\ ,\ \gamma^\mu\gamma^\nu\ ,\ \gamma^\mu\gamma^\nu\gamma^\rho\ ,\ \gamma^0\gamma^1\gamma^2\gamma^3\}_{\mu<\nu<\rho}. \tag{2.33}$$

Other (higher) products can be reduced to these using anticommutativity $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$, and the fact that $(\gamma^0)^2 = \mathbb{I}$, and $(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -\mathbb{I}$. We have in total $1+4+6+4+1 = 16$ matrices, which are linearly independent, as argued in Exercise 11 on the basis of the γ-matrix identities of Exercise 8. They generate the Clifford algebra of spacetime (in physics referred to as the Dirac algebra).

The identity matrix and its multiples represent numbers (scalars). Single matrices γ^μ represent an orthonormal basis of the Minkowski spacetime. Products of two are commonly presented as

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu,\gamma^\nu] \qquad (\text{for } \mu\neq\nu: \quad \sigma^{\mu\nu} = i\gamma^\mu\gamma^\nu), \tag{2.34}$$

and they can be visualized as planes spanned by the vectors γ^μ, γ^ν. Products of three γ-matrices represent three-dimensional spacetime hyperplanes. The last element, the product of all γ-matrices, is conventionally cast as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \text{so that} \quad (\gamma^5)^2 = 1, \quad \text{and it satisfies} \quad \gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu. \tag{2.35}$$

Let us now try to understand the role of the elements $\sigma^{\mu\nu}$ in the mathematical description of the Minkowski spacetime. First, observe a key identity

$$\frac{i}{2}[\sigma^{\mu\nu},\gamma^{\rho}] = -\frac{1}{4}[2g^{\mu\nu} - 2\gamma^{\nu}\gamma^{\mu},\gamma^{\rho}] = \frac{1}{2}\Big(\gamma^{\nu}\{\gamma^{\mu},\gamma^{\rho}\} - \{\gamma^{\nu},\gamma^{\rho}\}\gamma^{\mu}\Big) = g^{\mu\rho}\gamma^{\nu} - g^{\nu\rho}\gamma^{\mu}, \quad (2.36)$$

where we have used the 'anticommutator' Leibniz rule

$$[AB,C] = ABC + (ACB - ACB) - CAB = A\{B,C\} - \{A,C\}B. \quad (2.37)$$

Using the definition of the Lorentz generators $\mathsf{M}^{\mu\nu}$, Eq. (2.18), we can also write

$$\frac{i}{2}[\sigma^{\mu\nu},\gamma^{\rho}] = (g^{\mu\rho}\delta^{\nu}_{\sigma} - g^{\nu\rho}\delta^{\mu}_{\sigma})\gamma^{\sigma} = (-i\mathsf{M}^{\mu\nu})^{\rho}{}_{\sigma}\gamma^{\sigma}. \quad (2.38)$$

With a help of the Jacobi identity in the form $\big[A,[B,C]\big] = \big[[A,B],C\big] - \big[[A,C],B\big]$ we find

$$\begin{aligned}
[\sigma^{\mu\nu},\sigma^{\rho\sigma}] &= \frac{i}{2}\big[\sigma^{\mu\nu},[\gamma^{\rho},\gamma^{\sigma}]\big] \\
&= \frac{i}{2}\big[[\sigma^{\mu\nu},\gamma^{\rho}],\gamma^{\sigma}\big] - \frac{i}{2}\big[[\sigma^{\mu\nu},\gamma^{\sigma}],\gamma^{\rho}\big] \\
&= [g^{\mu\rho}\gamma^{\nu} - g^{\nu\rho}\gamma^{\mu},\gamma^{\sigma}] - [g^{\mu\sigma}\gamma^{\nu} - g^{\nu\sigma}\gamma^{\mu},\gamma^{\rho}] \\
&= -2i(g^{\mu\rho}\sigma^{\nu\sigma} - g^{\mu\sigma}\sigma^{\nu\rho} + g^{\nu\sigma}\sigma^{\mu\rho} - g^{\nu\rho}\sigma^{\mu\sigma}), \quad (2.39)
\end{aligned}$$

which shows that matrices $\frac{1}{2}\sigma^{\mu\nu}$ satisfy the same commutation relations as the Lorentz generators $\mathsf{M}^{\mu\nu}$, Eq. (2.22). This is a spacetime analogue of the relation between rotation generators M_i and $\frac{1}{2}\sigma^i$ in Sec. 2.1.1.

It is now natural to define the *spin representation*

$$S(L) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) \quad \text{of a Lorentz transformation} \quad \mathsf{L} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathsf{M}^{\mu\nu}\right). \quad (2.40)$$

The spacetime analogue of formula (2.7) then reads

$$S(L)^{-1}\gamma^{\mu}S(L) = L^{\mu}{}_{\nu}\gamma^{\nu}, \quad (2.41)$$

which can be proved using Eq. (2.38), and the Campbell identity (2.14) (analogously to Exercise 4). (Mind the inverse instead of Hermitian conjugation — the generators $\sigma^{\mu\nu}$ are actually not Hermitian, as we will see when adopting a concrete representation of the γ-matrices.)

With $S(L)$ we may not only transform four-vectors, but also consider an action of the form $S(L)\Psi$ on complex column vectors, referred to as the *Dirac spinors* — a spacetime analogue of the Pauli spinors.

2.3.1 Representations of γ-matrices

Explicit representations of γ^{μ}'s are found in the space of 4 by 4 matrices. (Smaller dimensionality would not allow for the 16 independent matrices in Eq. (2.33).) In Exercise 9 we show that the matrices

$$\gamma^0_D = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix} = \sigma^3 \otimes \mathbb{I} \quad , \quad \gamma^i_D = \begin{pmatrix} \mathbb{O} & \sigma^i \\ -\sigma^i & \mathbb{O} \end{pmatrix} = i\sigma^2 \otimes \sigma^i \quad (2.42)$$

that constitute the so-called Dirac (or 'standard') representation indeed satisfy the defining Clifford algebra relation (2.32). Here we have also displayed the tensor product notation, which often proves efficient when handling larger matrices.

The tensor (or Kronecker) product $\mathsf{A}\otimes\mathsf{B}$, of an $n\times n'$ matrix A, and an $m\times m'$ matrix B, is an $(nm)\times(n'm')$ matrix defined by

$$\mathsf{A}\otimes\mathsf{B}=\begin{pmatrix}A_{11}\mathsf{B} & \dots & A_{1n'}\mathsf{B}\\ \vdots & & \vdots\\ A_{n1}\mathsf{B} & \dots & A_{nn'}\mathsf{B}\end{pmatrix}. \tag{2.43}$$

This product is distributive and associative, but not commutative ($\mathsf{A}\otimes\mathsf{B}\neq\mathsf{B}\otimes\mathsf{A}$), and satisfies the fundamental identity

$$(\mathsf{A}\otimes\mathsf{B})(\mathsf{C}\otimes\mathsf{D})=(\mathsf{AC})\otimes(\mathsf{BD}). \tag{2.44}$$

Let us also note that $(\mathsf{A}\otimes\mathsf{B})^{-1}=\mathsf{A}^{-1}\otimes\mathsf{B}^{-1}$, and $(\mathsf{A}\otimes\mathsf{B})^{\dagger}=\mathsf{A}^{\dagger}\otimes\mathsf{B}^{\dagger}$.

Using this notation we can calculate, as an example,

$$\gamma_D^5=i\big((\sigma^3(i\sigma^2)^3\big)\otimes(\sigma^1\sigma^2\sigma^3)=(\sigma^3\sigma^2)\otimes(i\sigma^3\sigma^3)=\sigma^1\otimes\mathbb{I}=\begin{pmatrix}\mathbb{O} & \mathbb{I}\\ \mathbb{I} & \mathbb{O}\end{pmatrix}. \tag{2.45}$$

Under Hermitian conjugation the γ-matrices in Dirac representation behave as follow:

$$(\gamma_D^0)^\dagger=\gamma_D^0,\quad (\gamma_D^j)^\dagger=-\gamma_D^j=\gamma_D^0\gamma_D^j\gamma_D^0\quad\to\quad(\gamma_D^\mu)^\dagger=\gamma_D^0\gamma_D^\mu\gamma_D^0,\quad\text{and}\quad(\gamma_D^5)^\dagger=\gamma_D^5. \tag{2.46}$$

According to Pauli's fundamental theorem on γ-matrices (for a proof see [10, p. 132]) any other representation of γ-matrices can be obtained by a similarity transformation $\tilde{\gamma}^\mu=\mathsf{U}\gamma_D^\mu\mathsf{U}^{-1}$ (where U is unique up to a scalar factor). In addition, the matrix U is unitary if (and only if) the new representation $\tilde{\gamma}^\mu$ exhibits the same conjugation properties (2.46). The statement that a similarity transformation yields again a representation of the Clifford algebra is in fact easy to prove,

$$\{\tilde{\gamma}^\mu,\tilde{\gamma}^\nu\}=\{\mathsf{U}\gamma_D^\mu\mathsf{U}^{-1},\mathsf{U}\gamma_D^\nu\mathsf{U}^{-1}\}=\mathsf{U}\{\gamma_D^\mu,\gamma_D^\nu\}\mathsf{U}^{-1}=\mathsf{U}\,2g^{\mu\nu}\mathbb{I}\,\mathsf{U}^{-1}=2g^{\mu\nu}\mathbb{I}, \tag{2.47}$$

as well as the fact that unitary transformations preserve the conjugation properties.

As an important example, the Weyl (or 'chiral') representation is obtained as follows:

$$\gamma_W^\mu=\mathsf{U}\gamma_D^\mu\mathsf{U}^\dagger,\quad\text{where}\quad\mathsf{U}=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbb{I} & -\mathbb{I}\\ \mathbb{I} & \mathbb{I}\end{pmatrix}=\frac{1}{\sqrt{2}}(\mathbb{I}+\gamma_D^5\gamma_D^0)\quad\text{is unitary.} \tag{2.48}$$

Since $\mathsf{U}^\dagger=\frac{1}{\sqrt{2}}(\mathbb{I}+\gamma_D^0\gamma_D^5)$, and $\gamma^i\gamma^0\gamma^5=\gamma^0\gamma^5\gamma^i$, we find

$$\begin{aligned}\gamma_W^0&=\mathsf{U}\gamma_D^0\mathsf{U}^\dagger=\mathsf{U}\mathsf{U}^\dagger\gamma_D^5=\gamma_D^5,\\ \gamma_W^i&=\mathsf{U}\gamma_D^i\mathsf{U}^\dagger=\mathsf{U}\mathsf{U}^\dagger\gamma_D^i=\gamma_D^i,\\ \gamma_W^5&=i\gamma_D^5\gamma_D^1\gamma_D^2\gamma_D^3=\gamma_D^5\gamma_D^0\gamma_D^5=-\gamma_D^0.\end{aligned} \tag{2.49}$$

In Exercise 10 we show that the spin representation of the rotation generators reads explicitly

$$\frac{1}{4}\varepsilon_{ijk}\sigma_{D,W}^{jk}=\frac{1}{2}\Sigma^i,\quad\text{where}\quad\Sigma^i\equiv\begin{pmatrix}\sigma^i & \mathbb{O}\\ \mathbb{O} & \sigma^i\end{pmatrix}=\mathbb{I}\otimes\sigma^i, \tag{2.50}$$

and for the boost generators we obtain

$$\frac{1}{2}\sigma_D^{0i}=\frac{i}{2}\begin{pmatrix}\mathbb{O} & \sigma^i\\ \sigma^i & \mathbb{O}\end{pmatrix}=\frac{i}{2}\,\sigma^1\otimes\sigma^i\quad,\quad\frac{1}{2}\sigma_W^{0i}=\frac{i}{2}\begin{pmatrix}-\sigma^i & \mathbb{O}\\ \mathbb{O} & \sigma^i\end{pmatrix}=-\frac{i}{2}\,\sigma^3\otimes\sigma^i. \tag{2.51}$$

The spin representation of a finite Lorentz transformation determined by parameters $\boldsymbol{\theta}$ and $\boldsymbol{\zeta}$ reads (in the Dirac or Weyl representation, respectively)

$$S(L)_{D,W} = \exp\left(-\frac{i}{2}\theta^i\Sigma^i - \frac{i}{2}\zeta^i\sigma^{0i}_{D,W}\right). \tag{2.52}$$

Note that in the Weyl representation of γ-matrices the Lorentz generators coincide with $\mathsf{J}_i^{L\oplus R}$ and $\mathsf{K}_i^{L\oplus R}$ from Eq. (2.30). In fact, we do not need to introduce the algebra of γ-matrices to infer the spin representation of the Lorentz generators. Notwithstanding, we do need the γ-matrices in order to represent spacetime vectors by $\gamma^\mu a_\mu$, and Lorentz-transform them using the formula (2.41).

The notion of representation in the context of Clifford algebras should not be confused with the notion of representation in the context of Lie algebras. In the first case we look for matrices that satisfy the anticommutation rules $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ (and find, e.g., γ^μ_D or γ^μ_W), whereas in the second case we look for matrices that satisfy certain commutation relations $[\mathsf{A}_a, \mathsf{A}_b] = f_{abc}\mathsf{A}_c$ (and find, e.g., $\mathsf{M}^{\mu\nu}$ or $\frac{1}{2}\sigma^{\mu\nu}_{D,W}$).

2.4 Exercises

Exercise 4. Show that for $\alpha \in \mathbb{R}$ and a unit vector $\boldsymbol{n}$ the following identity holds:

$$\mathsf{U}^\dagger\,\sigma^i\,\mathsf{U} = R_{ij}\sigma^j, \quad \text{where} \quad \mathsf{U} = \exp\left(-\frac{i}{2}\alpha\,\boldsymbol{n}\cdot\boldsymbol{\sigma}\right) \quad \text{and} \quad \mathsf{R} = \exp(-i\alpha n^k \mathsf{M}_k). \tag{2.53}$$

Solution:

The identity is a simple consequence of the formula (2.14). From the commutator

$$\left[\frac{i}{2}\alpha\,\boldsymbol{n}\cdot\boldsymbol{\sigma}, \sigma^i\right] = \frac{i}{2}\alpha\,n^k\,2i\varepsilon_{kij}\sigma^j = -i\alpha\,n^k(\mathsf{M}_k)_{ij}\sigma^j \tag{2.54}$$

we find

$$\begin{aligned} e^{\frac{i}{2}\alpha\boldsymbol{n}\cdot\boldsymbol{\sigma}}\sigma^i e^{-\frac{i}{2}\alpha\boldsymbol{n}\cdot\boldsymbol{\sigma}} &= \sigma^i + (-i\alpha n^k\mathsf{M}_k)_{ij}\sigma^j + \frac{1}{2!}(-i\alpha n^k\mathsf{M}_k)_{ij}(-i\alpha n^\ell\mathsf{M}_\ell)_{jm}\sigma^m + \dots \\ &= (e^{-i\alpha n^k\mathsf{M}_k})_{ij}\sigma^j. \end{aligned} \tag{2.55}$$

Exercise 5. *Restricted Baker-Campbell-Hausdorff formula.* Show that

$$e^{\mathsf{A}}e^{\mathsf{B}} = e^{\mathsf{A}+\mathsf{B}}e^{\frac{1}{2}[\mathsf{A},\mathsf{B}]} \quad \text{provided} \quad [\mathsf{A},[\mathsf{A},\mathsf{B}]] = [\mathsf{B},[\mathsf{A},\mathsf{B}]] = 0. \tag{2.56}$$

Solution:

First we note that formula (2.14) for a matrix B satisfying $[\mathsf{A},[\mathsf{A},\mathsf{B}]] = 0$ reads simply

$$e^{\mathsf{A}}\,\mathsf{B}\,e^{-\mathsf{A}} = \mathsf{B} + [\mathsf{A},\mathsf{B}]. \tag{2.57}$$

Now consider a matrix-valued function

$$\mathsf{F}(s) = e^{s\mathsf{A}}e^{s\mathsf{B}}, \tag{2.58}$$

and calculate

$$\frac{d}{ds}\mathsf{F}(s) = e^{s\mathsf{A}}(\mathsf{A}+\mathsf{B})e^{s\mathsf{B}} = e^{s\mathsf{A}}(\mathsf{A}+\mathsf{B})e^{-s\mathsf{A}}\,\mathsf{F}(s) = \big(\mathsf{A}+\mathsf{B}+s[\mathsf{A},\mathsf{B}]\big)\,\mathsf{F}(s). \tag{2.59}$$

Integrating this differential equation, and taking into account the initial condition $\mathsf{F}(0) = \mathbb{I}$, we obtain

$$\mathsf{F}(s) = \exp\Big(s\mathsf{A} + s\mathsf{B} + \frac{s^2}{2}[\mathsf{A},\mathsf{B}]\Big), \tag{2.60}$$

which, when compared with the original definition (2.58), yields for $s = 1$ the desired formula.

Remarks:

For a generic matrix-valued function $\mathsf{A}(s)$,

$$\frac{d}{ds}e^{\mathsf{A}(s)} \neq \frac{d\mathsf{A}(s)}{ds}e^{\mathsf{A}(s)}, \tag{2.61}$$

since it is not guaranteed that $\mathsf{A}(s)$ and $\frac{d\mathsf{A}(s)}{ds}$ commute. If they do commute then equality holds. For example, for any scalar-valued function $\alpha(s)$ and a constant matrix A it does hold that

$$\frac{d}{ds}e^{\alpha(s)\mathsf{A}} = \frac{d\alpha(s)}{ds}\mathsf{A}\,e^{\alpha(s)\mathsf{A}}. \tag{2.62}$$

In our case, i.e., under the assumption (2.56), we can factorize $\mathsf{F}(s) = e^{s(\mathsf{A}+\mathsf{B})}e^{\frac{s^2}{2}[\mathsf{A},\mathsf{B}]}$, which can be differentiated using the Leibniz rule to verify that it indeed satisfies the differential equation (2.59).

Exercise 6. *Commutation relations of the Lorentz algebra.* Show that the matrices

$$(\mathsf{M}^{\mu\nu})^{\rho}{}_{\sigma} = i(g^{\mu\rho}\delta^{\nu}_{\sigma} - g^{\nu\rho}\delta^{\mu}_{\sigma}) \tag{2.63}$$

satisfy the commutation relations

$$[\mathsf{M}^{\mu\nu}, \mathsf{M}^{\rho\sigma}] = -i(g^{\mu\rho}\mathsf{M}^{\nu\sigma} - g^{\mu\sigma}\mathsf{M}^{\nu\rho} + g^{\nu\sigma}\mathsf{M}^{\mu\rho} - g^{\nu\rho}\mathsf{M}^{\mu\sigma}). \tag{2.64}$$

Solution:

In components we have

$$\begin{aligned}(\mathsf{M}^{\mu\nu})^{\kappa}{}_{\lambda}(\mathsf{M}^{\rho\sigma})^{\lambda}{}_{\tau} &= i^2(g^{\mu\kappa}\delta^{\nu}_{\lambda} - g^{\nu\kappa}\delta^{\mu}_{\lambda})(g^{\rho\lambda}\delta^{\sigma}_{\tau} - g^{\sigma\lambda}\delta^{\rho}_{\tau})\\ &= i^2(g^{\mu\kappa}g^{\rho\nu}\delta^{\sigma}_{\tau} - g^{\mu\kappa}g^{\sigma\nu}\delta^{\rho}_{\tau} - g^{\nu\kappa}g^{\rho\mu}\delta^{\sigma}_{\tau} + g^{\nu\kappa}g^{\sigma\mu}\delta^{\rho}_{\tau}),\\ (\mathsf{M}^{\rho\sigma})^{\kappa}{}_{\lambda}(\mathsf{M}^{\mu\nu})^{\lambda}{}_{\tau} &= i^2(g^{\rho\kappa}g^{\mu\sigma}\delta^{\nu}_{\tau} - g^{\rho\kappa}g^{\nu\sigma}\delta^{\mu}_{\tau} - g^{\sigma\kappa}g^{\mu\rho}\delta^{\nu}_{\tau} + g^{\sigma\kappa}g^{\nu\rho}\delta^{\mu}_{\tau}).\end{aligned} \tag{2.65}$$

Subtracting the two expressions yields

$$\begin{aligned}&[\mathsf{M}^{\mu\nu}, \mathsf{M}^{\rho\sigma}]^{\kappa}{}_{\tau}\\ &= i^2\Big(g^{\nu\rho}(g^{\mu\kappa}\delta^{\sigma}_{\tau} - g^{\sigma\kappa}\delta^{\mu}_{\tau}) - g^{\nu\sigma}(g^{\mu\kappa}\delta^{\rho}_{\tau} - g^{\rho\kappa}\delta^{\mu}_{\tau}) - g^{\mu\rho}(g^{\nu\kappa}\delta^{\sigma}_{\tau} - g^{\sigma\kappa}\delta^{\nu}_{\tau}) + g^{\mu\sigma}(g^{\nu\kappa}\delta^{\rho}_{\tau} - g^{\rho\kappa}\delta^{\nu}_{\tau})\Big)\\ &= i\Big(g^{\nu\rho}(\mathsf{M}^{\mu\sigma})^{\kappa}{}_{\tau} - g^{\nu\sigma}(\mathsf{M}^{\mu\rho})^{\kappa}{}_{\tau} - g^{\mu\rho}(\mathsf{M}^{\nu\sigma})^{\kappa}{}_{\tau} + g^{\mu\sigma}(\mathsf{M}^{\nu\rho})^{\kappa}{}_{\tau}\Big).\end{aligned} \tag{2.66}$$

Dropping the component indices, and reshuffling the terms, we obtain the desired result.

Exercise 7. *Lorentz boost.* Find explicit form of the Lorentz transformation matrix $\mathsf{L} = e^{-i\zeta\mathsf{K}_1}$.

Solution:

First we recall that from Eqs. (2.20) and (2.23)

$$(\mathsf{K}_1)^{\rho}{}_{\sigma} = (\mathsf{M}^{01})^{\rho}{}_{\sigma} = i(g^{0\rho}\delta^{1}_{\sigma} - g^{1\rho}\delta^{0}_{\sigma}) = i\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} = i\begin{pmatrix}\sigma^1 & \mathbb{O}\\ \mathbb{O} & \mathbb{O}\end{pmatrix} \tag{2.67}$$

Since functions on block-diagonal matrices act blockwise we have

$$\mathsf{L} = e^{-i\zeta\mathsf{K}_1} = \begin{pmatrix}e^{\zeta\sigma^1} & \mathbb{O}\\ \mathbb{O} & \mathbb{I}\end{pmatrix}, \tag{2.68}$$

and since $(\sigma^1)^2 = \mathbb{I}$ we find

$$e^{\zeta\sigma^1} = \cosh(\zeta\sigma^1) + \sinh(\zeta\sigma^1) = (\cosh\zeta)\mathbb{I} + (\sinh\zeta)\sigma^1 = \begin{pmatrix}\cosh\zeta & \sinh\zeta\\ \sinh\zeta & \cosh\zeta\end{pmatrix}, \tag{2.69}$$

where we have have realised that the Taylor series for the function cosh(.) only contains even powers, whereas the one for sinh(.) only contains odd powers of its argument.

Remarks:

The rapidity ζ is related to the boost velocity $\beta = v/c$ by $\tanh\zeta = \beta$, and to the relativistic γ-factor by $\cosh\zeta = \gamma$. The particle is being boosted by velocity β, i.e., the new coordinate system is moving with velocity $-\beta$ with respect to the original coordinate system.

Exercise 8. *Properties of γ-matrices.* Show that for 4 by 4 matrices the relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ implies

$$1.\quad \gamma^\mu\gamma_\mu = 4\,\mathbb{I} \tag{2.70}$$
$$2.\quad \gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu \tag{2.71}$$
$$3.\quad \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}\,\mathbb{I}, \tag{2.72}$$

and the 'trace identities'

$$4.\quad \mathrm{Tr}(\gamma^\mu) = 0 \tag{2.73}$$
$$5.\quad \mathrm{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}) = 0 \quad (\forall n = 0, 1, \dots) \tag{2.74}$$
$$6.\quad \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \tag{2.75}$$
$$7.\quad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{2.76}$$
$$8.\quad \mathrm{Tr}(\gamma^5) = 0 \tag{2.77}$$
$$9.\quad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0 \tag{2.78}$$
$$10.\quad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\varepsilon^{\mu\nu\rho\sigma}. \tag{2.79}$$

Solution:

1.
$$\gamma^\mu\gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = (\gamma^0)^2 - (\gamma^1)^2 - (\gamma^2)^2 - (\gamma^3)^2 = 4\,\mathbb{I}. \tag{2.80}$$

2. From $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2g^{\mu\nu}$,
$$\gamma^\mu\gamma^\nu\gamma_\mu = -\gamma^\nu\gamma^\mu\gamma_\mu + 2g^{\mu\nu}\gamma_\mu = -4\gamma^\nu + 2\gamma^\nu = -2\gamma^\nu. \tag{2.81}$$

3. Similarly,
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = -\gamma^\nu\gamma^\mu\gamma^\rho\gamma_\mu + 2g^{\mu\nu}\gamma^\rho\gamma_\mu = -\gamma^\nu(-2\gamma^\rho) + 2\gamma^\rho\gamma^\nu = 2\{\gamma^\nu, \gamma^\rho\} = 4g^{\nu\rho}\,\mathbb{I}. \tag{2.82}$$

4. Using $(\gamma^5)^2 = \mathbb{I}$, the anticommutativity $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$ $(\forall\mu)$, and cyclic property of the trace,
$$\mathrm{Tr}(\gamma^\mu) = \mathrm{Tr}(\gamma^5\gamma^5\gamma^\mu) = -\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^5) = -\mathrm{Tr}(\gamma^5\gamma^5\gamma^\mu) = -\mathrm{Tr}(\gamma^\mu) \quad\Rightarrow\quad \mathrm{Tr}(\gamma^\mu) = 0. \tag{2.83}$$

5. Similarly,
$$\mathrm{Tr}(\gamma^5\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}) = -\mathrm{Tr}(\gamma^5\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}\gamma^5) \quad\Rightarrow\quad \mathrm{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}) = 0. \tag{2.84}$$

6.
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\,\mathrm{Tr}(\{\gamma^\mu, \gamma^\nu\}) = \mathrm{Tr}(g^{\mu\nu}\,\mathbb{I}) = 4g^{\mu\nu}. \tag{2.85}$$

7. We repeatedly use the identity $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2g^{\mu\nu}$,

$$\begin{aligned}\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= -\mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma) + 2g^{\mu\nu}\,\mathrm{Tr}(\gamma^\rho\gamma^\sigma)\\ &= \mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma) - 2g^{\mu\rho}\,\mathrm{Tr}(\gamma^\nu\gamma^\sigma) + 2g^{\mu\nu}4g^{\rho\sigma}\\ &= -\mathrm{Tr}(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu) + 2g^{\mu\sigma}4g^{\mu\sigma} - 2g^{\mu\rho}4g^{\nu\sigma} + 2g^{\mu\nu}4g^{\rho\sigma},\end{aligned} \tag{2.86}$$

and cycle the γ^μ back to the front to find

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{2.87}$$

8. Since, $(\gamma^0)^2 = \mathbb{I}$, and $\gamma^0\gamma^5 = -\gamma^5\gamma^0$,

$$\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(\gamma^0\gamma^0\gamma^5) = -\mathrm{Tr}(\gamma^0\gamma^5\gamma^0) = -\mathrm{Tr}(\gamma^0\gamma^0\gamma^5) = -\mathrm{Tr}(\gamma^5) \quad \Rightarrow \quad \mathrm{Tr}(\gamma^5) = 0. \tag{2.88}$$

9. For $\mu = \nu$ reduces to Eq. (2.77). To illustrate the case $\mu \neq \nu$, let us consider, for concreteness, $\mu = 1$ and $\nu = 2$:

$$\mathrm{Tr}(\gamma^1\gamma^2\gamma^5) = i\,\mathrm{Tr}(\gamma^1\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3) = -i\,\mathrm{Tr}(\gamma^0\gamma^3) = 0 \tag{2.89}$$

by Eq. (2.75).

10. If any two indices are equal, we come back to Eq. (2.78). If all four indices differ, the γ-matrices anticommute, and can be reorder using the spacetime Levi-Civita symbol, yielding

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5) = -i\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(\gamma^5\gamma^5) = -4i\varepsilon^{\mu\nu\rho\sigma}. \tag{2.90}$$

Exercise 9. *Dirac representation of* γ*-matrices.* Verify that the matrices

$$\gamma^0_D = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix} = \sigma^3 \otimes \mathbb{I} \quad , \quad \gamma^i_D = \begin{pmatrix} \mathbb{O} & \sigma^i \\ -\sigma^i & \mathbb{O} \end{pmatrix} = i\sigma^2 \otimes \sigma^i \tag{2.91}$$

satisfy, for all μ, ν, the Clifford algebra relation $\{\gamma^\mu_D, \gamma^\nu_D\} = 2g^{\mu\nu}$.

Solution:

$$\begin{aligned}&\{\gamma^0_D, \gamma^0_D\} = 2(\gamma^0_D)^2 = 2\,\mathbb{I},\\ &\{\gamma^0_D, \gamma^i_D\} = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix}\begin{pmatrix} \mathbb{O} & \sigma^i \\ -\sigma^i & \mathbb{O} \end{pmatrix} + \begin{pmatrix} \mathbb{O} & \sigma^i \\ -\sigma^i & \mathbb{O} \end{pmatrix}\begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix} = \begin{pmatrix} \mathbb{O} & \sigma^i \\ \sigma^i & \mathbb{O} \end{pmatrix} + \begin{pmatrix} \mathbb{O} & -\sigma^i \\ -\sigma^i & \mathbb{O} \end{pmatrix} = \mathbb{O},\\ &\{\gamma^i_D, \gamma^j_D\} = (i\sigma^2)^2 \otimes (\sigma^i\sigma^j) + (i\sigma^2)^2 \otimes (\sigma^j\sigma^i) = -\mathbb{I} \otimes \{\sigma^i, \sigma^j\} = -2\delta_{ij}\mathbb{I}.\end{aligned} \tag{2.92}$$

Exercise 10. *Lorentz generators in Dirac and Weyl representation.* Find explicit matrix form of

$$1. \quad \frac{1}{4}\varepsilon_{ijk}\sigma^{jk}_{D,W} \qquad \text{and} \qquad 2. \quad \frac{1}{2}\sigma^{0i}_{D,W}. \tag{2.93}$$

Solution:

1. Since $\gamma^i_D = \gamma^i_W$, we can calculate at once

$$\frac{1}{4}\varepsilon_{ijk}\sigma^{jk}_{D,W} = \frac{i}{8}\varepsilon_{ijk}[\gamma^j_D, \gamma^k_D] = \frac{i}{4}\varepsilon_{ijk}\gamma^j_D\gamma^k_D = \frac{i}{4}\varepsilon_{ijk}(i\sigma^2)^2 \otimes (\sigma^j\sigma^k) = -\frac{i}{4}\varepsilon_{ijk}\mathbb{I}\otimes(i\varepsilon_{jk\ell}\sigma^\ell), \tag{2.94}$$

and as $\varepsilon_{ijk}\varepsilon_{jk\ell} = 2\delta_{i\ell}$, obtain

$$\frac{1}{4}\varepsilon_{ijk}\sigma^{jk}_{D,W} = \frac{1}{2}\delta_{i\ell}\,\mathbb{I}\otimes\sigma^\ell = \frac{1}{2}\mathbb{I}\otimes\sigma^i = \frac{1}{2}\begin{pmatrix}\sigma^i & \mathbb{O}\\ \mathbb{O} & \sigma^i\end{pmatrix}. \tag{2.95}$$

2.

$$\begin{aligned}\frac{1}{2}\sigma^{0i}_D &= \frac{i}{4}[\gamma^0_D, \gamma^i_D] = \frac{i}{2}\gamma^0_D\gamma^i_D = \frac{i}{2}\begin{pmatrix}\mathbb{I} & \mathbb{O}\\ \mathbb{O} & -\mathbb{I}\end{pmatrix}\begin{pmatrix}\mathbb{O} & \sigma^i\\ -\sigma^i & \mathbb{O}\end{pmatrix} = \frac{i}{2}\begin{pmatrix}\mathbb{O} & \sigma^i\\ \sigma^i & \mathbb{O}\end{pmatrix},\\ \frac{1}{2}\sigma^{0i}_W &= \frac{i}{2}\gamma^0_W\gamma^i_W = \frac{i}{2}\gamma^5_D\gamma^i_D = \frac{i}{2}\begin{pmatrix}\mathbb{O} & \mathbb{I}\\ \mathbb{I} & \mathbb{O}\end{pmatrix}\begin{pmatrix}\mathbb{O} & \sigma^i\\ -\sigma^i & \mathbb{O}\end{pmatrix} = \frac{i}{2}\begin{pmatrix}-\sigma^i & \mathbb{O}\\ \mathbb{O} & \sigma^i\end{pmatrix}.\end{aligned} \tag{2.96}$$

Exercise 11. *Basis in the space of 4 by 4 matrices.* Based on the results of Exercise 8 show that the matrices

$$\{\mathbb{I}\ ,\ \gamma^\mu\ ,\ \gamma^\mu\gamma^\nu\ ,\ \gamma^\mu\gamma^\nu\gamma^\rho\ ,\ \gamma^0\gamma^1\gamma^2\gamma^3\}_{\mu<\nu<\rho} \tag{2.97}$$

are linearly independent.

Solution:

First note that traces of all the elements apart from $\mathbb{I}$ vanish:

$$\mathrm{Tr}(\gamma^\mu) = \mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = \mathrm{Tr}(\gamma^0\gamma^1\gamma^2\gamma^3) = 0, \quad \text{provided} \quad \mu<\nu<\rho, \tag{2.98}$$

due to the trace identities (2.73), (2.75), (2.74) and (2.77).

Now set to zero an arbitrary (complex) linear combination:

$$\alpha_0\mathbb{I} + \alpha_\mu\gamma^\mu + \alpha_{\mu\nu}\gamma^\mu\gamma^\nu + \alpha_{\mu\nu\rho}\gamma^\mu\gamma^\nu\gamma^\rho + \alpha_{0123}\gamma^0\gamma^1\gamma^2\gamma^3 = 0, \tag{2.99}$$

where the summations run only over $\mu<\nu<\rho$. Taking the trace yields

$$\alpha_0\,\mathrm{Tr}(\mathbb{I}) = 4\alpha_0 = 0 \quad\Rightarrow\quad \alpha_0 = 0. \tag{2.100}$$

Any other α-coefficient can be rendered zero by first multiplying Eq. (2.99) by the corresponding γ-monomial, and then taking the trace. For example, choosing α_{01} we have

$$\alpha_0\gamma^0\gamma^1 + \alpha_\mu\gamma^0\gamma^1\gamma^\mu + \alpha_{\mu\nu}\gamma^0\gamma^1\gamma^\mu\gamma^\nu + \alpha_{\mu\nu\rho}\gamma^0\gamma^1\gamma^\mu\gamma^\nu\gamma^\rho + \alpha_{0123}\gamma^2\gamma^3 = \ldots + \alpha_{01}\mathbb{I} + \ldots = 0, \tag{2.101}$$

where '...' contain nonzero power of γ's, which vanish upon tracing.

That is, vanishing of the linear combination (2.99) implies vanishing of all its coefficients, hence the 16 matrices (2.97) are linearly independent, and for 4 by 4 matrices form a basis of $\mathbb{C}^{4,4}$.

Chapter 3

Dirac theory – basics

3.1 Dirac equation and its Lorentz covariance

With a help of the γ-matrices satisfying the anticommutation rules of the Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, Dirac was able to famously factorize the Klein-Gordon operator (see Section 1.3.3), writing the Klein-Gordon equation in the form

$$(i\partial^\mu i\partial_\mu - m^2)\Psi = (i\gamma^\mu\partial_\mu + m)(i\gamma^\nu\partial_\nu - m)\Psi = 0, \tag{3.1}$$

and postulate a stronger condition on the wave-function Ψ — the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0. \tag{3.2}$$

(The sign before m can be flipped by a change of Clifford algebra representation $\gamma^\mu \to -\gamma^\mu$.) This equation is sometimes written as

$$(i\gamma\cdot\partial - m)\Psi = 0 \quad \text{or} \quad (i\not\partial - m)\Psi = 0. \tag{3.3}$$

The former form is engraved on Dirac's memorial tile in Westminster Abbey. The latter one uses Feynman's slash notation $\not a \equiv \gamma^\mu a_\mu$, which is quite common in physics, but we shall not employ it too much in these lectures.

By default, we will adopt the standard representation of γ-matrices given explicitly by the 4 by 4 matrices from Eq. (2.42). $\Psi = (\psi_\alpha)_{\alpha=1}^4$ is therefore a four-component (Dirac) wave-function. For brevity we will drop the subscript D, and simply write γ^μ instead of γ^μ_D. In a different representation $\tilde\gamma^\mu = \mathsf{U}\gamma^\mu\mathsf{U}^{-1}$ we have

$$0 = \mathsf{U}(i\gamma^\mu\partial_\mu - m)\Psi(x) = (i\mathsf{U}\gamma^\mu\mathsf{U}^{-1}\partial_\mu - m)\mathsf{U}\Psi(x) = (i\tilde\gamma^\mu\partial_\mu - m)\tilde\Psi(x), \quad \text{where} \quad \tilde\Psi(x) = \mathsf{U}\Psi(x) \tag{3.4}$$

is a wave-function in the new representation.

What are the transformation properties of the Dirac equation under Lorentz transformations $x'^\mu = L^\mu{}_\nu x^\nu$? The differential operator $\gamma^\mu\partial_\mu$ (also known as the Dirac operator) transforms as

$$\gamma^\mu\partial'_\mu = \gamma^\mu\frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = (\mathsf{L}^{-1})^\nu{}_\mu\gamma^\mu\,\partial_\nu = S(L)\gamma^\nu S(L)^{-1}\,\partial_\nu \tag{3.5}$$

where we used relation (2.41) for L^{-1}, and the fact that the spin representation satisfies $S(L^{-1}) = S(L)^{-1}$. Hence, we find that the Dirac equation transforms covariantly as

$$(i\gamma^\mu\partial'_\mu - m)\Psi'(x') = S(L)(i\gamma^\nu\partial_\nu - m)\underbrace{S(L)^{-1}\Psi'(x'(x))}_{\Psi(x)} = 0, \tag{3.6}$$

once we identify the Lorentz-transformed Dirac wave-function with

$$\Psi'(x') = S(L)\Psi(x). \tag{3.7}$$

(Here we consider only proper orthochronous Lorentz transformations. Discrete transformations, parity and time reversal, will be addressed in Sec. 4.2.) We observe that for rotations, according to Eq. (2.52),

$$S(L) = \exp\left(-\frac{i}{2}\theta^i\Sigma^i\right) = \begin{pmatrix} \exp\left(-\frac{i}{2}\theta^i\sigma^i\right) & \mathbb{O} \\ \mathbb{O} & \exp\left(-\frac{i}{2}\theta^i\sigma^i\right) \end{pmatrix}, \tag{3.8}$$

both the upper and the lower half of Ψ transform as two-component Pauli spinors — the Dirac theory describes spin-$\frac{1}{2}$ particles, and the four-component wave-function Ψ is dubbed the *Dirac spinor* (or *bispinor*).

Hermitian conjugation of the transformation law (3.7) produces

$$\big(\Psi'(x')\big)^\dagger = \big(\Psi(x)\big)^\dagger S(L)^\dagger, \quad \text{with} \quad S(L) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right). \tag{3.9}$$

Since

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \quad \to \quad (\sigma^{\mu\nu})^\dagger = -\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)^\dagger = -\frac{i}{2}\gamma^0(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)\gamma^0 = \gamma^0\sigma^{\mu\nu}\gamma^0, \tag{3.10}$$

the Lorentz generators $\frac{1}{2}\sigma^{\mu\nu}$ are Hermitian for rotations, but anti-Hermitian for boosts. As a result, the spin representation of Lorentz transformations is not unitary, but rather satisfies

$$S(L)^\dagger = \exp\left(\frac{i}{4}\omega_{\mu\nu}\gamma^0\sigma^{\mu\nu}\gamma^0\right) = \gamma^0 S(L)^{-1}\gamma^0, \tag{3.11}$$

i.e., $S(L)$ is unitary for rotations, but for boosts it is Hermitian.

In the Dirac theory, more applicable than Hermitian conjugation turns out to be its slight modification — the *Dirac conjugation*

$$\bar{\Psi}(x) \equiv \Psi^\dagger(x)\gamma^0. \tag{3.12}$$

The transformation law for Dirac-conjugated spinors reads

$$\bar{\Psi}'(x') = \big(\Psi(x)\big)^\dagger S(L)^\dagger\gamma^0 = \bar{\Psi}(x)S(L)^{-1}, \tag{3.13}$$

and the equation satisfied by $\bar{\Psi}(x)$ can be derived from the Dirac equation as follows:

$$\begin{aligned} i\gamma^\mu\partial_\mu\Psi - m\Psi &= 0 \quad / \ ^\dagger \\ -i\partial_\mu\Psi^\dagger\gamma^0\gamma^\mu\gamma^0 - m\Psi^\dagger &= 0 \quad / \ \gamma^0 \\ i\partial_\mu\bar{\Psi}\gamma^\mu + m\bar{\Psi} &= 0. \end{aligned} \tag{3.14}$$

3.2 Plane wave solutions

The Dirac equation (3.2) is a linear differential equation with constant coefficients (albeit with certain matrix structure), and so we look for solutions in the form of plane waves (with certain polarization states). From Eq. (3.1) it is clear that these must also satisfy the relativistic energy-momentum dispersion relation $p^\mu p_\mu = (p_0)^2 - \boldsymbol{p}^2 = m^2$. To take into account both positive and negative energies, it is common to distinguish two types of plane waves parametrized by the spatial momentum $\boldsymbol{p}$:

$$\Psi^{(+)}_{\boldsymbol{p}}(x) = u(\boldsymbol{p})e^{-ip\cdot x},$$
$$\Psi^{(-)}_{\boldsymbol{p}}(x) = v(\boldsymbol{p})e^{+ip\cdot x}, \quad \text{where in both cases} \quad p_0 = \omega_{\boldsymbol{p}} \equiv \sqrt{\boldsymbol{p}^2 + m^2}. \tag{3.15}$$

They satisfy $i\partial_t \Psi^{(+)}_{\boldsymbol{p}} = \omega_{\boldsymbol{p}}\Psi^{(+)}_{\boldsymbol{p}}$, so $\Psi^{(+)}_{\boldsymbol{p}}$ is a *positive-energy* solution, while $i\partial_t \Psi^{(-)}_{\boldsymbol{p}} = -\omega_{\boldsymbol{p}}\Psi^{(-)}_{\boldsymbol{p}}$, so $\Psi^{(-)}_{\boldsymbol{p}}$ is a *negative-energy* solution.

With the plane wave ansatz the Dirac equation reduces to an algebraic relation for the polarization spinor $u(\boldsymbol{p})$ (for positive energy), or $v(\boldsymbol{p})$ (negative energy):

$$(\gamma^\mu p_\mu - m)u(\boldsymbol{p}) = 0, \quad \text{or} \quad (\gamma^\mu p_\mu + m)v(\boldsymbol{p}) = 0, \tag{3.16}$$

respectively.

We first investigate the positive-energy equation in particle's rest frame (hence assuming $m > 0$). In this case the four-momentum reads $p^{(0)}_\mu = (m, \boldsymbol{0})$, $\gamma^\mu p^{(0)}_\mu = \gamma^0 m$, with $\gamma^0 = \sigma^3 \otimes \mathbb{I}$, and Eq. (3.16) reduces to

$$(\gamma^0 - 1)u(\boldsymbol{0}) = -2\begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix} u(\boldsymbol{0}) = 0. \tag{3.17}$$

This tells that the lower components of $u(\boldsymbol{0})$ vanish, and so we get two linearly independent solutions, characterized by the value of spin projection along the z-direction:

$$u(\boldsymbol{0}, s) = \begin{pmatrix} \chi_s \\ 0 \\ 0 \end{pmatrix}, \quad \text{where} \quad s = \frac{1}{2}, -\frac{1}{2}, \quad \text{and} \quad \chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.18}$$

The spin sum

$$\sum_s u(\boldsymbol{0}, s)\bar{u}(\boldsymbol{0}, s) = \sum_s u(\boldsymbol{0}, s)u^T(\boldsymbol{0}, s)\gamma^0 = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} = \frac{1}{2}(\gamma^0 + 1) \tag{3.19}$$

is the rest-frame projector on states with positive energy.

A positive-energy solution with an arbitrary spatial momentum $\boldsymbol{p}$ is obtained from a rest-frame solution $u(\boldsymbol{0})$ by boosting. Denoting the boosted four-momentum by $p_\nu = L_\nu{}^\mu p^{(0)}_\mu$, we observe that

$$0 = S(L)(\gamma^\mu p^{(0)}_\mu - m)S(L)^{-1}S(L)u(\boldsymbol{0}) = (\gamma^\nu L_\nu{}^\mu p^{(0)}_\mu - m)S(L)u(\boldsymbol{0}) = (\gamma^\mu p_\mu - m)S(L)u(\boldsymbol{0}), \tag{3.20}$$

so

$$u(\boldsymbol{p}) = S(L)u(\boldsymbol{0}), \qquad \text{where} \qquad S(L) = \frac{\gamma^\mu\gamma^0 p_\mu + m}{\sqrt{2m}\sqrt{p_0 + m}} = \frac{\sqrt{p_0 + m}}{\sqrt{2m}}\begin{pmatrix} \mathbb{I} & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p_0+m} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p_0+m} & \mathbb{I} \end{pmatrix} \tag{3.21}$$

is the boost's spin representation, as shown in Exercise 12. Two independent positive-energy polarization spinors (for arbitrary $\boldsymbol{p}$) then read

$$u(\boldsymbol{p}, s) = S(L)u(\boldsymbol{0}, s) = \frac{\sqrt{p_0 + m}}{\sqrt{2m}}\begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p_0+m}\chi_s \end{pmatrix}. \tag{3.22}$$

Since Dirac conjugation of these yields

$$\bar{u}(\boldsymbol{p},s)=\big(S(L)u(\mathbf{0},s)\big)^{\dagger}\gamma^{0}=u^{\dagger}(\mathbf{0},s)S(L)^{\dagger}\gamma^{0}=\bar{u}(\mathbf{0},s)S(L)^{-1},\tag{3.23}$$

we find from Eq. (3.19) the spin sum (i.e., the projector on positive-energy states) for arbitrary $\boldsymbol{p}$,

$$\sum_s u(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s)=S(L)\sum_s u(\mathbf{0},s)\bar{u}(\mathbf{0},s)S(L)^{-1}=\frac{1}{2m}S(L)(\gamma^0 m+m)S(L)^{-1}=\frac{\gamma^\mu p_\mu+m}{2m}.\tag{3.24}$$

The negative-energy plane waves can be treated similarly. We find again two independent polarization spinors for each value of $\boldsymbol{p}$:

$$v(\mathbf{0},s)=\begin{pmatrix}0\\0\\\chi_s\end{pmatrix}\quad\to\quad v(\boldsymbol{p},s)=S(L)v(\mathbf{0},s)=\frac{\sqrt{p_0+m}}{\sqrt{2m}}\begin{pmatrix}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p_0+m}\chi_s\\\chi_s\end{pmatrix}.\tag{3.25}$$

The spin sum for negative-energy states reads, for $\boldsymbol{p}=\mathbf{0}$,

$$\sum_s v(\mathbf{0},s)\bar{v}(\mathbf{0},s)=\sum_s v(\mathbf{0},s)v^T(\mathbf{0},s)\gamma^0=-\begin{pmatrix}\mathbb{O}&\mathbb{O}\\\mathbb{O}&\mathbb{I}\end{pmatrix}=\frac{1}{2}(\gamma^0-1),\tag{3.26}$$

and for $\boldsymbol{p}$ arbitrary

$$\sum_s v(\boldsymbol{p},s)\bar{v}(\boldsymbol{p},s)=\frac{\gamma^\mu p_\mu-m}{2m}.\tag{3.27}$$

Utilizing both the u and v spin sums we can form (for every fixed $\boldsymbol{p}$) a completeness relation on the spinor space,

$$\sum_s\Big(u(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s)-v(\boldsymbol{p},s)\bar{v}(\boldsymbol{p},s)\Big)=\mathbb{I},\tag{3.28}$$

where the u-part is the projector on the positive energy subspace, while the v-part (including the minus sign) is the projector on the negative energy subspace.

To summarize, the general solution of the Dirac equation is given by the plane wave decomposition

$$\Psi(x)=\sum_{s=\pm\frac{1}{2}}\int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{\omega_{\boldsymbol{p}}}}\Big(B_{\boldsymbol{p},s}u(\boldsymbol{p},s)e^{-ip\cdot x}+D^*_{\boldsymbol{p},s}v(\boldsymbol{p},s)e^{ip\cdot x}\Big)\quad,\quad p_0=\omega_{\boldsymbol{p}}=\sqrt{\boldsymbol{p}^2+m^2},\tag{3.29}$$

where $B_{\boldsymbol{p},s}$ and $D_{\boldsymbol{p},s}$ are arbitrary complex constants (amplitudes of the corresponding plane waves), and the numerical factor $\sqrt{m/(2\pi)^3\omega_{\boldsymbol{p}}}$ has been included for later convenience. The negative-energy part is related to antiparticles. Although Dirac proposed the existence of antiparticles based on his theory of relativistic quantum mechanics, we shall postpone their discussion until the quantum field theory part of the course.

Finally, let us mention that the polarisation spinors u and v satisfy the identities

$$\bar{u}(\boldsymbol{p},s)u(\boldsymbol{p},s')=\delta_{ss'}\quad,\quad\bar{v}(\boldsymbol{p},s)v(\boldsymbol{p},s')=-\delta_{ss'}\quad,\quad\bar{u}(\boldsymbol{p},s)v(\boldsymbol{p},s')=0,\tag{3.30}$$

which for $\boldsymbol{p}=\mathbf{0}$ follow easily from Eqs. (3.18) and (3.25) (and for generic $\boldsymbol{p}$ by utilizing $u(\boldsymbol{p},s)=S(L)u(\mathbf{0},s)$, $\bar{u}(\boldsymbol{p},s)=\bar{u}(\mathbf{0},s)S(L)^{-1}$, and likewise for v), and also the identities (derived in Exercise 14)

$$u^\dagger(\boldsymbol{p},s)u(\boldsymbol{p},s')=\frac{\omega_{\boldsymbol{p}}}{m}\delta_{ss'}\quad,\quad v^\dagger(\boldsymbol{p},s)v(\boldsymbol{p},s')=\frac{\omega_{\boldsymbol{p}}}{m}\delta_{ss'}\quad,\quad u^\dagger(\boldsymbol{p},s)v(-\boldsymbol{p},s')=0,\tag{3.31}$$

which will be used later in Chapter 8 when deriving the mode decomposition of the Hamiltonian operator of the quantised Dirac field.

3.3 Dirac bilinears

The γ-matrices and the Dirac spinors live in a rather abstract space. To relate them to the real world (i.e., the spacetime) we construct out of them the so-called *Dirac bilinears* — certain number-valued quadratic expressions in Ψ with spacetime indices.

The most common one is the Dirac current

$$J^\mu(x) = \bar{\Psi}(x)\gamma^\mu\Psi(x). \tag{3.32}$$

It transforms as a four-vector field under Lorentz transformations (recall Eqs. (3.7) and (3.13)):

$$J'^\mu(x') = \bar{\Psi}'(x')\gamma^\mu\Psi'(x') = \bar{\Psi}(x)S(L)^{-1}\gamma^\mu S(L)\Psi(x) = \bar{\Psi}(x)L^\mu{}_\nu\gamma^\nu\Psi(x) = L^\mu{}_\nu J^\nu(x). \tag{3.33}$$

Its zeroth component reads

$$J^0(x) = \Psi^\dagger(x)\Psi(x) \geq 0, \tag{3.34}$$

and it gives, upon spatial integration, a positive-definite norm of the Dirac wave-functions.

The Dirac current J^μ is always conserved if $\Psi(x)$ satisfies the Dirac equation $(i\gamma^\mu\partial_\mu - m)\Psi = 0$ (and hence $\bar{\Psi}(x)$ satisfies its conjugate version $i\partial_\mu\bar{\Psi}\gamma^\mu + m\bar{\Psi} = 0$):

$$\partial_\mu J^\mu = (\partial_\mu\bar{\Psi})\gamma^\mu\Psi + \bar{\Psi}\gamma^\mu\partial_\mu\Psi = im\bar{\Psi}\Psi + \bar{\Psi}(-im)\Psi = 0. \tag{3.35}$$

When the Dirac field gets coupled to the electromagnetic field, J^μ will become the electric current density that features on the right-hand side of Maxwell's equation (see Eq. (11.51)).

General tensor fields can be constructed analogously as $\bar{\Psi}(x)\gamma^\mu \ldots \gamma^\nu\Psi(x)$. Indeed, the transformation properties of such objects are

$$\begin{aligned}\bar{\Psi}'(x')\gamma^\mu \ldots \gamma^\nu\Psi'(x') &= \bar{\Psi}(x)S(L)^{-1}\gamma^\mu S(L) \ldots S(L)^{-1}\gamma^\nu S(L)\Psi(x) \\ &= L^\mu{}_\rho \ldots L^\nu{}_\sigma\bar{\Psi}(x)\gamma^\rho \ldots \gamma^\sigma\Psi(x).\end{aligned} \tag{3.36}$$

(This also includes the scalar field $\bar{\Psi}(x)\Psi(x)$.)

Pseudotensors contain an extra factor $\det \mathsf{L}$ in their transformation law. They exhibit their "pseudo" character under parity L_P, for which $\det \mathsf{L}_P = -1$. In Section 4.2.2 we will show that the spin representation of the parity transformation is $S(L_P) = \gamma^0$. Hence,

$$S(L_P)^{-1}\gamma^5 S(L_P) = \gamma^0\gamma^5\gamma^0 = -\gamma^5, \quad \text{whereas} \quad S(L)^{-1}\gamma^5 S(L) = \gamma^5 \tag{3.37}$$

for all proper Lorentz transformations L, since $\gamma^5\sigma^{\mu\nu} = \sigma^{\mu\nu}\gamma^5$. Therefore, we can form pseudotensors by including γ^5 in the string $\gamma^\mu \ldots \gamma^\nu$ in Eq. (3.36). For example, $\bar{\Psi}\gamma^5\Psi$ is a pseudoscalar, $\bar{\Psi}\gamma^\mu\gamma^5\Psi$ a pseudovector (or axial vector), etc.

3.4 Exercises

Exercise 12. *Boosting a particle from the rest.* Consider four-momentum $p^{(0)}_\mu \equiv (m, \mathbf{0})$ (particle at rest), and find the spin representation $S(L)$ of the Lorentz boost that transforms the particle's three-momentum from $\mathbf{0}$ to an arbitrary given vector $\boldsymbol{p}$.

Solution:

We will present two solution methods.

1. We need to boost the particle by velocity
$$\boldsymbol{v} = \frac{\boldsymbol{p}}{p_0}, \quad \text{i.e., with rapidity} \quad \boldsymbol{\zeta} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|}\zeta, \quad \text{where} \quad \tanh\zeta = |\boldsymbol{v}|. \tag{3.38}$$
Observing that
$$(\zeta^i\gamma^0\gamma^i)^2 = -\zeta^i\zeta^j\gamma^i\gamma^j = -\zeta^i\zeta^j\frac{1}{2}\{\gamma^i,\gamma^j\} = \zeta^i\zeta^i = \zeta^2, \tag{3.39}$$
we find
$$S(L) = \exp\left(-\frac{i}{2}\zeta^i\sigma^{0i}\right) = e^{\frac{1}{2}\zeta^i\gamma^0\gamma^i} = \cosh\frac{\zeta}{2} + \frac{\zeta^i}{\zeta}\gamma^0\gamma^i\sinh\frac{\zeta}{2} = \cosh\frac{\zeta}{2} + \frac{p^i}{|\boldsymbol{p}|}\gamma^0\gamma^i\sinh\frac{\zeta}{2}. \tag{3.40}$$
Finally, since $\cosh\zeta = \gamma = \frac{p_0}{m}$, and employing identities for hyperbolic functions
$$\cosh\frac{\zeta}{2} = \sqrt{\frac{\cosh\zeta+1}{2}} = \sqrt{\frac{p_0+m}{2m}} \quad , \quad \sinh\frac{\zeta}{2} = \sqrt{\frac{\cosh\zeta-1}{2}} = \sqrt{\frac{p_0-m}{2m}}, \tag{3.41}$$
which follow from $(e^{\zeta/2} \pm e^{-\zeta/2})^2 = e^\zeta + e^{-\zeta} \pm 2$, we can cast the result in a more convenient form
$$S(L) = \sqrt{\frac{p_0+m}{2m}} + \frac{p^i}{|\boldsymbol{p}|}\gamma^0\gamma^i\sqrt{\frac{p_0-m}{2m}} = \frac{p_0+m+\frac{p^i}{|\boldsymbol{p}|}\gamma^0\gamma^i\sqrt{p_0^2-m^2}}{\sqrt{2m}\sqrt{p_0+m}} = \frac{\gamma^\mu p_\mu\gamma^0+m}{\sqrt{2m}\sqrt{p_0+m}}. \tag{3.42}$$

2. Alternatively, we can employ the identity (2.41) for L^{-1}:
$$S(L)\gamma^\mu S(L)^{-1} = (\mathsf{L}^{-1})^\mu{}_\nu\gamma^\nu = \gamma^\nu L_\nu{}^\mu \quad \rightarrow \quad S(L)\gamma^\mu p^{(0)}_\mu S(L)^{-1} = \gamma^\nu L_\nu{}^\mu p^{(0)}_\mu = \gamma^\nu p_\nu, \tag{3.43}$$
where $p_\mu = L_\mu{}^\nu p^{(0)}_\nu$. Since L is a boost,
$$\sigma^{0i} = i\gamma^0\gamma^i \quad \rightarrow \quad \gamma^0 S(L)^{-1} = \gamma^0\exp\left(\frac{i}{2}\zeta^i\sigma^{0i}\right) = \exp\left(-\frac{i}{2}\zeta^i\sigma^{0i}\right)\gamma^0 = S(L)\gamma^0, \tag{3.44}$$
and so we have
$$S(L)\gamma^\mu p^{(0)}_\mu S(L)^{-1} = S(L)\gamma^0 m S(L)^{-1} = S(L)^2\gamma^0 m = \gamma^\mu p_\mu \quad \rightarrow \quad S(L)^2 = \frac{1}{m}\gamma^\mu p_\mu\gamma^0. \tag{3.45}$$
Hence, we look for a square root of the matrix $\frac{1}{m}\gamma^\mu p_\mu\gamma^0$. This can be found by first calculating the square
$$(\gamma^\mu p_\mu\gamma^0)^2 = \gamma^\mu p_\mu\gamma^0\gamma^\nu p_\nu\gamma^0 = -\gamma^\mu\gamma^\nu p_\nu p_\mu + 2g^{0\nu}\gamma^\mu p_\mu p_\nu\gamma^0 = -p^\mu p_\mu + 2p_0\gamma^\mu p_\mu\gamma^0. \tag{3.46}$$

Rearranging the terms, noting that $p^\mu p_\mu = m^2$, and completing the square yields

$$\begin{aligned}(\gamma^\mu p_\mu \gamma^0)^2 \pm 2m\gamma^\mu p_\mu \gamma^0 + m^2 &= 2p_0\gamma^\mu p_\mu \gamma^0 \pm 2m\gamma^\mu p_\mu \gamma^0 \\ \frac{1}{m}\gamma^\mu p_\mu \gamma^0 &= \frac{(\gamma^\mu p_\mu \gamma^0 \pm m)^2}{2m(p_0 \pm m)}. \end{aligned} \tag{3.47}$$

Hence

$$S(L) = \frac{\gamma^\mu p_\mu \gamma^0 + m}{\sqrt{2m}\sqrt{p_0 + m}}, \tag{3.48}$$

where we have kept only the '+' sign as the '−' option does not reduce to 1 in the limit $\boldsymbol{p} \to \boldsymbol{0}$.

Exercise 13. *Gordon decomposition.* Derive the (Gordon) identities

$$\begin{aligned}\bar{u}(\boldsymbol{p}')\gamma^\mu u(\boldsymbol{p}) &= \bar{u}(\boldsymbol{p}')\left(\frac{p'^\mu + p^\mu}{2m} + i\sigma^{\mu\nu}\frac{p'_\nu - p_\nu}{2m}\right)u(\boldsymbol{p}), \\ -\bar{v}(\boldsymbol{p}')\gamma^\mu v(\boldsymbol{p}) &= \bar{v}(\boldsymbol{p}')\left(\frac{p'^\mu + p^\mu}{2m} + i\sigma^{\mu\nu}\frac{p'_\nu - p_\nu}{2m}\right)v(\boldsymbol{p}), \end{aligned} \tag{3.49}$$

which hold for any polarization spinors $u(\boldsymbol{p})$, $u(\boldsymbol{p}')$, $v(\boldsymbol{p})$, $v(\boldsymbol{p}')$ satisfying the equations

$$(\gamma^\mu p_\mu - m)u(\boldsymbol{p}) = 0 \quad , \quad (\gamma^\mu p_\mu + m)v(\boldsymbol{p}) = 0. \tag{3.50}$$

Solution:

The Dirac equation for u and for its conjugate, $\bar{u}(\boldsymbol{p}')(\gamma^\mu p'_\mu - m) = 0$, imply

$$\begin{aligned}\bar{u}(\boldsymbol{p}')(\gamma^\nu\gamma^\mu p_\mu - m\gamma^\nu)u(\boldsymbol{p}) &= 0, \\ \bar{u}(\boldsymbol{p}')(\gamma^\mu\gamma^\nu p'_\mu - m\gamma^\nu)u(\boldsymbol{p}) &= 0 \end{aligned} \tag{3.51}$$

Summing these two equations, and using the decomposition

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = g^{\mu\nu} - i\sigma^{\mu\nu} \tag{3.52}$$

to cast

$$\gamma^\nu\gamma^\mu p_\mu + \gamma^\mu\gamma^\nu p'_\mu = p^\nu - i\sigma^{\nu\mu}p_\mu + p'^\nu - i\sigma^{\mu\nu}p'_\mu, \tag{3.53}$$

we obtain

$$\bar{u}(\boldsymbol{p}')\Big(p'^\nu + p^\nu + i\sigma^{\nu\mu}(p'_\mu - p_\mu)\Big)u(\boldsymbol{p}) = \bar{u}(\boldsymbol{p}')(2m\gamma^\nu)u(\boldsymbol{p}), \tag{3.54}$$

which yields the first identity in Eq. (3.49).

In the same manner we obtain the second identity (with u replaced by v, and m replaced by $-m$).

Exercise 14. *Polarisation spinors' identities.* Show that:

$$\begin{aligned} &1. \quad u^\dagger(\boldsymbol{p}, s)u(\boldsymbol{p}, s') = \frac{\omega_{\boldsymbol{p}}}{m}\delta_{ss'} \\ &2. \quad v^\dagger(\boldsymbol{p}, s)v(\boldsymbol{p}, s') = \frac{\omega_{\boldsymbol{p}}}{m}\delta_{ss'} \\ &3. \quad u^\dagger(\boldsymbol{p}, s)v(-\boldsymbol{p}, s') = 0. \end{aligned} \tag{3.55}$$

Solution:

1. By Eq. (3.49) for $\mu = 0$, and $p' = p$,

$$u^\dagger(\boldsymbol{p}, s)u(\boldsymbol{p}, s') = \bar{u}(\boldsymbol{p}, s)\gamma^0 u(\boldsymbol{p}, s') = \bar{u}(\boldsymbol{p}, s)\,\frac{p^0}{m}\,u(\boldsymbol{p}, s') = \frac{\omega_{\boldsymbol{p}}}{m}\delta_{ss'}, \tag{3.56}$$

   where in the last step we used the orthogonality relations (3.30).

2. Same as case 1. (Note that the minus sign in the Gordon identity for v gets cancelled with the minus in Eq. (3.30).)

3. We recall that for an appropriate boost L, $u(\boldsymbol{p}, s) = S(L)u(\boldsymbol{0}, s)$ (see Eq. (3.22)), hence $v(-\boldsymbol{p}, s') = S(L)^{-1}v(\boldsymbol{0}, s')$. Noting that, for a boost, $\gamma^0 S(L)^{-1} = S(L)\gamma^0$, we find

$$u^\dagger(\boldsymbol{p}, s)v(-\boldsymbol{p}, s') = \bar{u}(\boldsymbol{0}, s)S(L)^{-1}\gamma^0 S(L)^{-1}v(\boldsymbol{0}, s') = \bar{u}(\boldsymbol{0}, s)\gamma^0 v(\boldsymbol{0}, s') = u^\dagger(\boldsymbol{0}, s)v(\boldsymbol{0}, s') = 0. \tag{3.57}$$

Exercise 15. *Dirac equation in Weyl representation.* Write the Dirac equation in the Weyl representation of γ-matrices,

$$(i\gamma^\mu_W \partial_\mu - m)\Psi_W = 0, \quad \text{where} \quad \Psi_W = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{3.58}$$

as a pair of equations for ψ_L and ψ_R. Consider the limit $m \to 0$ corresponding to high energy and momentum (fast particles).

Solution:

In the Weyl representation,

$$i\gamma^\mu_W \partial_\mu = i\begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}\partial_0 + i\begin{pmatrix} \mathbb{O} & \sigma^i \\ -\sigma^i & \mathbb{O} \end{pmatrix}\partial_i, \tag{3.59}$$

so the Dirac equation reads

$$i\begin{pmatrix} \mathbb{O} & \mathbb{I}\,\partial_0 + \sigma^i\partial_i \\ \mathbb{I}\,\partial_0 - \sigma^i\partial_i & \mathbb{O} \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = m\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{3.60}$$

For massless particles, or in the limit of negligible rest mass m, i.e., dominant kinetic energy (fast particles), we obtain two independent equations

$$i\underbrace{(\mathbb{I}\,\partial_0 + \sigma^i\partial_i)}_{\equiv\sigma^\mu\partial_\mu}\psi_R = 0 \quad , \quad i\underbrace{(\mathbb{I}\,\partial_0 - \sigma^i\partial_i)}_{\equiv\bar{\sigma}^\mu\partial_\mu}\psi_L = 0, \tag{3.61}$$

called the right-handed, and the left-handed *Weyl equation*, respectively.

Chapter 4

Dirac theory – further developments

4.1 Dirac particle in electromagnetic field

In classical Hamiltonian mechanics, placing a free particle with mass m and charge q into an external electromagnetic field consists in the replacement

$$H = \frac{\boldsymbol{p}^2}{2m} \quad \to \quad H - qA_0 = \frac{(\boldsymbol{p} - q\boldsymbol{A})^2}{2m}. \tag{4.1}$$

Here, the electromagnetic field is described by a four-potential $A^\mu(x) = (A^0(x), A^i(x))$, from which one can calculate the Faraday tensor, and the electric and magnetic intensities:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad , \quad E_i = -E^i = F^{0i} \quad , \quad B_i = -B^i = \frac{1}{2}\varepsilon_{ijk}F^{jk}. \tag{4.2}$$

In quantum theory, analogously, the four-momentum operator $\hat{p}_\mu = i\partial_\mu$ is modified:

$$\hat{p}_\mu \quad \to \quad \hat{p}_\mu - qA_\mu, \qquad \text{i.e.,} \qquad \partial_\mu \quad \to \quad D_\mu \equiv \partial_\mu + iqA_\mu. \tag{4.3}$$

Here D_μ is the *covariant derivative*, and the replacement $\partial_\mu \to D_\mu$ is referred to as the *minimal coupling*. The minimal coupling procedure generalizes beyond electromagnetism to so-called non-Abelian gauge theories (or Yang-Mills theories) that play an important role in the Standard model of particle physics.

The Dirac equation for a charged particle in external electromagnetic field is thus obtained from the free Dirac equation by the replacement

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \quad \to \quad (i\gamma^\mu D_\mu - m)\Psi(x) = 0. \tag{4.4}$$

Another way of rewriting this is by multiplying γ^0 on the left, and thus casting the minimally coupled Dirac equation in the Schrödinger-like form

$$i\partial_t\Psi = \hat{H}_D\Psi, \quad \text{where} \quad \hat{H}_D = -i\gamma^0\gamma^i D_i + qA_0(x) + m\gamma^0 = \begin{pmatrix} qA_0 + m & -i\sigma^i D_i \\ -i\sigma^i D_i & qA_0 - m \end{pmatrix} \tag{4.5}$$

is the *Dirac Hamiltonian*, and we have used the Dirac representation of γ-matrices. This will be our starting point in the investigation of the non-relativistic limit of the Dirac equation in Section 4.1.1.

In Section 4.1.2 we shall make use of the following consequence of Eq. (4.4). Applying the operator $i\gamma^\mu D_\mu + m$ we have

$$(i\gamma^\mu D_\mu + m)(i\gamma^\nu D_\nu - m)\Psi = (-\gamma^\mu\gamma^\nu D_\mu D_\nu - m^2)\Psi = 0, \tag{4.6}$$

which can be cast, using the identities

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = g^{\mu\nu} - i\sigma^{\mu\nu}, \tag{4.7}$$

and

$$[D_\mu, D_\nu] = [\partial_\mu + iqA_\mu, \partial_\nu + iqA_\nu] = iq(\partial_\mu A_\nu - \partial_\nu A_\mu) = iqF_{\mu\nu}, \tag{4.8}$$

as

$$\Big(D^\mu D_\mu - i\sigma^{\mu\nu}\frac{1}{2}[D_\mu, D_\nu] + m^2\Big)\Psi(x) = \Big(D^\mu D_\mu + \frac{q}{2}\sigma^{\mu\nu}F_{\mu\nu} + m^2\Big)\Psi(x) = 0. \tag{4.9}$$

(We have used the antisymmetry of $\sigma^{\mu\nu}$ to write $\sigma^{\mu\nu}D_\mu D_\nu = \frac{1}{2}\sigma^{\mu\nu}[D_\mu, D_\nu]$.) Without the middle 'spin' term this would be a minimally coupled Klein-Gordon equation.

4.1.1 Non-relativistic limit of Dirac equation

As for the Klein-Gordon equation in Exercise 2, we pull the fast-oscillating rest-energy factor e^{-imt} out of the wave-function, and plug the following ansatz into Eq. (4.5):

$$\Psi(x) = e^{-imt}\begin{pmatrix}\varphi(x)\\ \eta(x)\end{pmatrix}: \quad (i\partial_t + m)\begin{pmatrix}\varphi\\ \eta\end{pmatrix} = \begin{pmatrix} qA_0 + m & -i\sigma^i D_i\\ -i\sigma^i D_i & qA_0 - m\end{pmatrix}\begin{pmatrix}\varphi\\ \eta\end{pmatrix}. \tag{4.10}$$

Two coupled equations for the two-component wave-functions η and φ ensue:

$$\begin{aligned} i\partial_t\varphi &= qA_0\varphi - i\sigma^i D_i\eta,\\ i\partial_t\eta + 2m\eta &= -i\sigma^i D_i\varphi + qA_0\eta. \end{aligned} \tag{4.11}$$

In the non-relativistic regime, and for weak electric fields, we can neglect in the second equation the terms with $i\partial_t\eta$ and $qA_0\eta$ against the dominant rest-energy term $2m\eta$, and obtain $2m\eta = -i\sigma^i D_i\varphi$. Expressing η and plugging it into the first equation, we find, in this approximation,

$$\eta = -\frac{i}{2m}\sigma^i D_i\varphi \quad \to \quad i\partial_t\varphi = -\frac{1}{2m}\sigma^i\sigma^j D_i D_j\varphi + qA_0\varphi. \tag{4.12}$$

Note that in full units we have $\eta = -\frac{i\hbar}{2mc}\sigma^i D_i\varphi$, which is of order $\frac{|\boldsymbol{v}|}{c}$, so in the non-relativistic limit the Dirac spinor's 'lower component' η is much smaller than the 'upper component' φ. This is consistent with the fact that at rest ($\boldsymbol{p} = \boldsymbol{0}$), the lower components of the polarization spinors u vanish (recall Eq. (3.18)).

Meanwhile we calculate, taking into account Eqs. (4.8) and (4.2),

$$\sigma^i\sigma^j D_i D_j\varphi = (\delta_{ij} + i\varepsilon_{ijk}\sigma^k)D_i D_j\varphi = D_i D_i\varphi + \frac{i}{2}\varepsilon_{ijk}\sigma^k\, iqF_{ij}\varphi = D_i D_i\varphi + q\sigma^k B^k\varphi, \tag{4.13}$$

and find the non-relativistic Pauli equation for a spin-$\frac{1}{2}$ particle with charge q described by a two-component wave-function $\varphi(\boldsymbol{x}, t)$,

$$i\partial_t\varphi = -\frac{1}{2m}D_i D_i\varphi - \frac{q}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}\,\varphi + qA_0\varphi. \tag{4.14}$$

The middle term corresponds to the magnetic interaction energy

$$-\boldsymbol{\mu}\cdot\boldsymbol{B}, \quad \text{where the magnetic moment} \quad \boldsymbol{\mu} = \frac{q}{m}\frac{\boldsymbol{\sigma}}{2} = \frac{2q}{|e|}\frac{|e|}{2m}\boldsymbol{S}. \tag{4.15}$$

Here $\boldsymbol{S} = \frac{\boldsymbol{\sigma}}{2}$ is the spin operator of a spin-$\frac{1}{2}$ particle, $|e|$ is the elementary charge, and the factor $g = \frac{2q}{|e|}$ is the so-called *g-factor*. For an electron ($q = -|e|$) the Dirac theory therefore provides the value $g_e = -2$. It should be emphasized that this is a *consequence* of the theory, unlike in non-relativistic quantum mechanics where $g_e = -2$ was established so as to fit experimental data.

In fact, current experimental value of the electron's g-factor is $g_e \doteq -2.00232$ [11, p. 16]. The small correction, called the *anomalous magnetic moment of electron*, is understood in the realm of quantum electrodynamics, where it was found by Schwinger that (in the first perturbative order)

$$\frac{|g_e| - 2}{2} = \frac{\alpha}{2\pi}, \quad \text{where} \quad \alpha \equiv \frac{e^2}{4\pi\varepsilon_0\hbar c} \doteq \frac{1}{137} \tag{4.16}$$

is the dimensionless *fine structure constant*.

4.1.2 Coulomb potential

Now we will investigate a Dirac particle, namely, an electron with charge $-|e|$, in a static electric field of an atomic nucleus with proton number Z. The electromagnetic four-potential is of the form $A_\mu(x) = (A_0(r), \mathbf{0})$, where $r \equiv |\boldsymbol{x}|$, and the potential energy reads

$$V(r) = -|e|A_0(r) = -|e|\frac{Z|e|}{4\pi\varepsilon_0}\frac{1}{r} = -\frac{Z\alpha}{r}. \tag{4.17}$$

Non-relativistic quantum mechanics provides the energy spectrum

$$E_N^{(NR)} = -\frac{mZ^2\alpha^2}{2N^2}, \quad \text{where} \quad N = n + \ell + 1 \quad (n, \ell = 0, 1, 2, \ldots). \tag{4.18}$$

From this we can estimate electron's typical velocity in the N-th state, v_N. Using the Feynman-Hellmann relation we find

$$\langle \frac{\hat{\boldsymbol{p}}^2}{2m} + \hat{V} \rangle_N = E_N^{(NR)} \quad \xrightarrow{\alpha\frac{\partial}{\partial\alpha}} \quad \langle \hat{V} \rangle_N = 2E_N^{(NR)}, \tag{4.19}$$

which allows us to express

$$\frac{\langle \hat{\boldsymbol{p}}^2 \rangle_N}{2m} = -E_N^{(NR)} = \frac{mZ^2\alpha^2}{2N^2} \quad \rightarrow \quad v_N \equiv \sqrt{\frac{\langle \hat{\boldsymbol{p}}^2 \rangle_N}{m^2}} = \frac{Z\alpha}{N}. \tag{4.20}$$

We can therefore expect that relativistic corrections to electron's energy levels will be significant especially for atoms with large proton number Z.

In order to find the energy spectrum within the Dirac theory it is convenient to start from Eq. (4.9). To proceed we evaluate

$$\begin{aligned} D^\mu D_\mu &= (\partial_t - i|e|A_0)(\partial_t - i|e|A_0) + \partial^i\partial_i = \left(\partial_t - i\frac{Z\alpha}{r}\right)^2 - \Delta, \\ \sigma^{\mu\nu}F_{\mu\nu} &= i\gamma^\mu\gamma^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) = 2i\gamma^i\gamma^0\partial_i A_0, \end{aligned} \tag{4.21}$$

and for stationary states $\Psi(\boldsymbol{x},t) = e^{-itE}\Psi_E(\boldsymbol{x})$ obtain

$$\begin{aligned} &\left[\Big(-iE - i\frac{Z\alpha}{r}\Big)^2 - \Delta - |e| i\gamma^i\gamma^0\partial_i A_0 + m^2\right]\Psi_E = 0 \\ \to\quad &\left[-E^2 - \frac{2Z\alpha E}{r} - \frac{Z^2\alpha^2}{r^2} - \Delta - i\gamma^i\gamma^0\partial_i\frac{Z\alpha}{r} + m^2\right]\Psi_E = 0 \\ \to\quad &\left[-\Delta - \frac{Z^2\alpha^2}{r^2} + i\gamma^i\gamma^0\frac{Z\alpha x^i}{r^3} - \frac{2Z\alpha E}{r} - (E^2 - m^2)\right]\Psi_E = 0. \end{aligned} \tag{4.22}$$

Note that to arrive at this point we did not need to make use of any concrete representation of γ-matrices. We are therefore free to adopt, for convenience, the Weyl representation, in which

$$\gamma^i\gamma^0 = \begin{pmatrix} \mathbb{O} & \sigma^i \\ -\sigma^i & \mathbb{O} \end{pmatrix}\begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix} = \begin{pmatrix} \sigma^i & \mathbb{O} \\ \mathbb{O} & -\sigma^i \end{pmatrix}. \tag{4.23}$$

Splitting the four-component Dirac wave-function into two parts, and casting the Laplace operator in spherical coordinates,

$$\Psi_E = \begin{pmatrix} \varphi_E \\ \eta_E \end{pmatrix} \quad , \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\hat{L}^2}{r^2}, \tag{4.24}$$

leads to the equation

$$\left[-\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\Big(\hat{L}^2 - Z^2\alpha^2 + iZ\alpha\frac{\sigma^i x^i}{r}\Big) - \frac{2Z\alpha E}{r} - (E^2 - m^2)\right]\varphi_E = 0, \tag{4.25}$$

and a similar equation for η_E, with $-i$ instead of $+i$. This difference however does not exhibit itself on the level of energy eigenvalues as we argue in Exercise 17, to where we shall relegate the rest of the calculation.

The resulting energy levels are

$$E_{N,j} = m + E_N^{(NR)}\left(1 + \frac{Z^2\alpha^2}{N(j+\frac{1}{2})} - \frac{3}{4}\frac{Z^2\alpha^2}{N^2} + \mathcal{O}(Z^4\alpha^4)\right), \quad \text{where} \quad j = \frac{1}{2}, \frac{3}{2}, \ldots, N - \frac{1}{2} \tag{4.26}$$

corresponds to the total (orbital plus spin) angular momentum operator $\hat{J}^i = \hat{L}^i + \frac{1}{2}\Sigma^i$. It is worth to note that for the (spinless) Klein-Gordon equation, j would be replaced by the orbital quantum number $\ell = 0, 1, \ldots, N-1$ [3, p. 73], resulting in a wrong prediction for the electron energy levels. In Exercise 16 we show that for a spinning particle it is the total angular momentum $\hat{J}^i$ that commutes with the Dirac Hamiltonian $\hat{H}_D$ (hence is conserved and corresponds to a quantum number), whereas the orbital part $\hat{L}^i$ itself does not.

Further corrections to formula (4.26) are due to the *hyperfine structure*, which takes into account the spin of the atomic nucleus; and the *Lamb shift*, which takes into account the quantum nature of the electromagnetic field.

4.2 Discrete transformations C, P and T

4.2.1 Charge conjugation

Let us write the Dirac equation in an external electromagnetic field in the form

$$[i\gamma^\mu(\partial_\mu + iqA_\mu) - m]\Psi = 0. \tag{4.27}$$

The sign of the electric charge q can be flipped by taking complex conjugation:

$$[-i(\gamma^\mu)^*(\partial_\mu - iqA_\mu) - m]\Psi^* = 0. \tag{4.28}$$

The matrices $-(\gamma^\mu)^*$ form a representation of the Clifford algebra,

$$\{-(\gamma^\mu)^*, -(\gamma^\nu)^*\} = \{\gamma^\mu, \gamma^\nu\}^* = (2g^{\mu\nu})^* = 2g^{\mu\nu}, \tag{4.29}$$

and so there exists a similarity transformation U such that

$$-(\gamma^\mu)^* = \mathsf{U}^{-1}\gamma^\mu\mathsf{U}. \tag{4.30}$$

For definiteness let us assume Dirac representation for the γ^μ's, in which γ^0, γ^1 and γ^3 are real matrices, whereas γ^2 is purely imaginary. Hence we find

$$-\gamma^{0,1,3} = \mathsf{U}^{-1}\gamma^{0,1,3}\,\mathsf{U} \quad , \quad \gamma^2 = \mathsf{U}^{-1}\gamma^2\mathsf{U} \qquad \to \qquad \mathsf{U} = \gamma^2 \tag{4.31}$$

up to an arbitrary phase factor.

Equation (4.28) now takes the form

$$[i\,\mathsf{U}^{-1}\gamma^\mu\mathsf{U}(\partial_\mu - iqA_\mu) - m]\Psi^* = \mathsf{U}^{-1}[i\gamma^\mu(\partial_\mu - iqA_\mu) - m]\Psi_C = 0, \tag{4.32}$$

where

$$\Psi_C(x) = \gamma^2\Psi^*(x) \tag{4.33}$$

is the charge-conjugated Dirac wave-function. If Ψ is a wave-function of an electron then Ψ_C is a wave-function of a particle with opposite charge but equal mass, namely, the positron.

Finally, let us note that it is common to define the charge conjugation operator $C = \gamma^2\gamma^0$, so that $\Psi_C = C\gamma^0\Psi^*$.

4.2.2 Parity

Space reflection, or parity,

$$x' = (x^0, -\boldsymbol{x}) \quad , \quad \mathsf{L}_P = \mathrm{diag}(1, -1, -1, -1) \tag{4.34}$$

is an improper Lorentz transformation ($\det \mathsf{L}_P = -1$). The spin representation of this transformation is determined from the requirement $S(L_P)^{-1}\gamma^\mu S(L_P) = (\mathsf{L}_P)^\mu{}_\nu\gamma^\nu$, i.e.,

$$S(L_P)^{-1}\gamma^0 S(L_P) = \gamma^0 \quad \text{and} \quad S(L_P)^{-1}\gamma^i S(L_P) = -\gamma^i \quad \Rightarrow \quad S(L_P) = \gamma^0 \tag{4.35}$$

up to an undetermined phase factor.

It is easy to check that if $(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0$, the parity-transformed wave-function

$$\Psi_P(x^0, -\boldsymbol{x}) = \gamma^0\Psi(x) \tag{4.36}$$

satisfies the Dirac equation in the new coordinates,

$$(i\gamma^\mu\partial'_\mu - m)\Psi_P(x') = (i\gamma^0\partial_0 - i\gamma^i\partial_i - m)\gamma^0\Psi(x) = \gamma^0(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\Psi(x) = 0. \tag{4.37}$$

It is worth to note that in the Weyl representation, parity interchanges left- and right-handed components of the Dirac spinor:

$$\gamma^0_W = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix} \quad \to \quad S(L_P)\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}. \tag{4.38}$$

4.2.3 Time reversal

Time reversal

$$x' = (-x^0, \boldsymbol{x}) \quad , \quad \mathsf{L}_T = \mathrm{diag}(-1, 1, 1, 1) \tag{4.39}$$

is an improper non-orthochronous Lorentz transformation. From quantum mechanics we know that it is implemented by an antiunitary operator. Hence we look for a transformed wave-function in the form

$$\Psi_T(-x^0, \boldsymbol{x}) = \mathsf{U}\Psi^*(x), \tag{4.40}$$

so that for appropriate U, $(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0$ implies

$$(i\gamma^\mu\partial'_\mu - m)\Psi_T(x') = (-i\gamma^0\partial_0 + i\gamma^1\partial_1 + i\gamma^2\partial_2 + i\gamma^3\partial_3 - m)\mathsf{U}\Psi^*(x) = 0. \tag{4.41}$$

Complex conjugating the original Dirac equation gives, assuming the Dirac representation,

$$(-i\gamma^0\partial_0 - i\gamma^1\partial_1 + i\gamma^2\partial_2 - i\gamma^3\partial_3 - m)\Psi^*(x) = 0. \tag{4.42}$$

Clearly, we need to meet the conditions

$$\gamma^{0,2}\mathsf{U} = \mathsf{U}\gamma^{0,2} \quad , \quad \gamma^{1,3}\mathsf{U} = -\mathsf{U}\gamma^{1,3}, \tag{4.43}$$

which is achieved with

$$\mathsf{U} = \gamma^1\gamma^3 \tag{4.44}$$

(up to an arbitrary phase).

The wave-function after time reversal therefore reads

$$\Psi_T(-x^0, \boldsymbol{x}) = \gamma^1\gamma^3\Psi^*(x). \tag{4.45}$$

4.2.4 CPT transformation

What is the wave-function after successive application of C, P and T transformations? Combining Eqs. (4.33), (4.36) and (4.45) we find

$$((\Psi_C)_P)_T(x) = \gamma^1\gamma^3\left(\gamma^0\gamma^2\Psi^*(-x)\right)^* = -i\gamma^5\Psi(-x). \tag{4.46}$$

The CPT *theorem* states that any local Lorentz invariant quantum field theory is necessarily invariant under the combined CPT transformation. However, experiments show that the individual transformations are not always symmetries of Nature. For example, parity is violated in the Standard model of electroweak interactions.

4.3 Helicity and chirality

The free-particle Dirac Hamiltonian

$$\hat{H}_D = \gamma^0\gamma^i\hat{p}^i + m\gamma^0 \tag{4.47}$$

does not commute with the spin operators $\frac{1}{2}\Sigma^i$ (see Eq. (4.63)). However, it does commute, as shown in Exercise 16, with their projection to the direction of motion — the *helicity*

$$\hat{h} \equiv \frac{\boldsymbol{\Sigma}}{2} \cdot \frac{\hat{\boldsymbol{p}}}{|\hat{\boldsymbol{p}}|}. \tag{4.48}$$

Since commuting with the Hamiltonian, and thus conserved during time evolution, helicity is a convenient quantity to characterise the spin state of the Dirac particle. However, it is not a Lorentz invariant concept due to its dependence on the spatial momentum $\boldsymbol{p}$.

A simple calculation reveals that

$$\hat{h}^2 = \frac{1}{4}\mathbb{I} \otimes \frac{(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}})^2}{|\hat{\boldsymbol{p}}|^2} = \frac{1}{4}, \tag{4.49}$$

therefore the eigenvalues of $\hat{h}$ are $\pm\frac{1}{2}$ (as must be the case for a spin projection). States with helicity $-\frac{1}{2}$ are referred to as left-handed, and those with $+\frac{1}{2}$ as right-handed.

Chirality is another notion of 'handedness'. It is represented by the operator γ^5, and since $(\gamma^5)^2 = 1$, it has eigenvalues -1, for left-chiral states, or $+1$, for right-chiral states. (One often uses the terms 'left-handed' and 'right-handed', which however increases the risk of confusing chirality with helicity.) Unlike helicity, chirality is not a constant of motion,

$$[\gamma^5, \hat{H}_D] = [\gamma^5, \gamma^0\gamma^i\hat{p}^i + m\gamma^0] = 2m\gamma^5\gamma^0 \neq 0, \tag{4.50}$$

unless $m = 0$. On the other hand, chirality is a concept invariant under proper Lorentz transformations, $S(L)\gamma^5 S(L)^{-1} = \gamma^5$, and gets flipped by parity, $S(L_P)\gamma^5 S(L_P)^{-1} = -\gamma^5$ (recall that $S(L_P) = \gamma^0$).

The *chiral projectors* $\frac{1}{2}(1 \pm \gamma^5)$, satisfying

$$\left(\frac{1 \pm \gamma^5}{2}\right)^2 = \frac{1 \pm \gamma^5}{2} \quad , \quad \frac{1+\gamma^5}{2} + \frac{1-\gamma^5}{2} = 1 \quad , \quad \frac{1+\gamma^5}{2}\frac{1-\gamma^5}{2} = 0, \tag{4.51}$$

can be used to decompose any state Ψ to its left-chiral (the "$-$" sign) and right-chiral (the "$+$" sign) components:

$$\gamma^5 \left(\frac{1 \pm \gamma^5}{2}\Psi\right) = \pm\left(\frac{1 \pm \gamma^5}{2}\Psi\right). \tag{4.52}$$

(Similarly, since $(2\hat{h})^2 = 1$, we could form helicity projectors $\frac{1}{2}(1 \pm 2\hat{h})$.) In the Weyl representation, which is also called 'chiral' as it is well-suited for discussing chirality, we have

$$\gamma^5_W = \begin{pmatrix} -\mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix} \quad \rightarrow \quad \frac{1-\gamma^5_W}{2} = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \quad , \quad \frac{1+\gamma^5_W}{2} = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix}. \tag{4.53}$$

Therefore, left-chiral bispinors are of the form $\begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$, whereas right-handed bispinors are of the form $\begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$.

As a final remark, consider a plane-wave solution of the *massless* Dirac equation ($p^\mu p_\mu = 0$), with polarization spinor $u(\boldsymbol{p})$ satisfying

$$\gamma^\mu p_\mu u(\boldsymbol{p}) = 0 \quad \rightarrow \quad u(\boldsymbol{p}) = \frac{\gamma^0\gamma^i p^i}{|\boldsymbol{p}|}u(\boldsymbol{p}) \tag{4.54}$$

Multiplying on the left by the chirality operator γ^5, and observing that

$$\gamma^5\gamma^0\gamma^i = -i\gamma^1\gamma^2\gamma^3\gamma^i = \frac{i}{2}\varepsilon_{ijk}\gamma^j\gamma^k = \frac{1}{2}\varepsilon_{ijk}\sigma^{jk} = \Sigma^i, \tag{4.55}$$

we find

$$\gamma^5 u(\boldsymbol{p}) = \frac{\Sigma^i p^i}{|\boldsymbol{p}|}u(\boldsymbol{p}) = 2h\, u(\boldsymbol{p}) \tag{4.56}$$

That is, for massless particles (or in the ultra-relativistic limit $m \to 0$), helicity and chirality coincide.

4.4 Exercises

Exercise 16. *Dirac Hamiltonian commutators.*

Show that for $A_\mu(x) = (A_0(r), \mathbf{0})$ (particle in central electric field) the Dirac Hamiltonian $\hat{H}_D$ commutes with the total angular momentum

$$\hat{J}^i = \hat{L}^i + \frac{\Sigma^i}{2}, \quad \text{where} \quad \hat{L}^i = \varepsilon_{ijk} x^j \hat{p}^k, \quad \Sigma^i = \begin{pmatrix} \sigma^i & \mathbb{O} \\ \mathbb{O} & \sigma^i \end{pmatrix}. \tag{4.57}$$

If in addition $A_0 = 0$ (free particle), show that $\hat{H}_D$ commutes with the helicity operator

$$\hat{h} = \frac{\boldsymbol{\Sigma}}{2} \cdot \frac{\hat{\boldsymbol{p}}}{|\hat{\boldsymbol{p}}|}. \tag{4.58}$$

Solution:

According to Eq. (4.5) the Dirac Hamiltonian reads

$$\hat{H}_D = -i\gamma^0\gamma^i\partial_i + qA_0(r) + m\gamma^0 = \gamma^0\gamma^i\hat{p}^i + V(r) + m\gamma^0. \tag{4.59}$$

Observe that expanding the commutator $[\hat{J}^i, \hat{H}_D]$ into six terms, two of them, namely, $[\hat{L}^i, \gamma^0]$ and $[\Sigma^i, V(r)]$ vanish trivially, $[\Sigma^i, \gamma^0]$ vanishes since Σ^i is a linear combination of products of two spatial γ-matrices, and

$$[\hat{L}^i, V(r)] = \varepsilon_{ijk}[x^j\hat{p}^k, V(r)] = -i\varepsilon_{ijk}x^j[\partial_k, V(r)] = -i\varepsilon_{ijk}x^j\frac{x^k}{r}V'(r) = 0. \tag{4.60}$$

We are thus left with the orbital angular momentum part

$$[\hat{L}^i, \hat{H}_D] = [\hat{L}^i, \gamma^0\gamma^j\hat{p}^j] = \gamma^0\gamma^j[\hat{L}^i, \hat{p}^j], \tag{4.61}$$

where

$$[\hat{L}^i, \hat{p}^j] = \varepsilon_{ik\ell}[x^k\hat{p}^\ell, \hat{p}^j] = \varepsilon_{ik\ell}[x^k, \hat{p}^j]\,\hat{p}^\ell = i\varepsilon_{ij\ell}\,\hat{p}^\ell, \tag{4.62}$$

and the spin angular momentum part

$$\frac{1}{2}[\Sigma^i, \hat{H}_D] = \frac{1}{2}[\Sigma^i, \gamma^0\gamma^j\hat{p}^j] = \frac{1}{2}\gamma^0[\Sigma^i, \gamma^j]\,\hat{p}^j, \tag{4.63}$$

where

$$[\Sigma^i, \gamma^j] = [\mathbb{I}\otimes\sigma^i, i\sigma^2\otimes\sigma^j] = i\sigma^2\otimes[\sigma^i, \sigma^j] = i\sigma^2\otimes(2i\varepsilon_{ijk}\sigma^k) = 2i\varepsilon_{ijk}\gamma^k. \tag{4.64}$$

Thus, in total,

$$[\hat{J}^i, \hat{H}_D] = [\hat{L}^i, \hat{H}_D] + \frac{1}{2}[\Sigma^i, \hat{H}_D] = i\gamma^0\gamma^j\varepsilon_{ij\ell}\,\hat{p}^\ell + i\gamma^0\gamma^k\varepsilon_{ijk}\,\hat{p}^j = 0. \tag{4.65}$$

For $A_0 = 0$, $\hat{p}^i$ commutes with $\hat{H}_D$ and so we find

$$[\hat{h}, \hat{H}_D] = \frac{\hat{p}^i}{|\hat{\boldsymbol{p}}|}\frac{1}{2}[\Sigma^i, \hat{H}_D] = \frac{\hat{p}^i}{|\hat{\boldsymbol{p}}|}i\gamma^0\gamma^k\varepsilon_{ijk}\,\hat{p}^j = 0. \tag{4.66}$$

Exercise 17. *Energy levels in Coulomb potential.* Starting from equation

$$\Big[-\Big(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\Big)+\frac{1}{r^2}\Big(\hat{L}^2-Z^2\alpha^2+iZ\alpha\frac{\sigma^i x^i}{r}\Big)-\frac{2Z\alpha E}{r}-(E^2-m^2)\Big]\varphi_E=0, \tag{4.67}$$

determine the energy levels of a Dirac electron in the Coulomb potential.

Solution:

Our strategy will be to use of the knowledge of non-relativistic energy spectrum

$$E_N^{(NR)}=-\frac{mZ^2\alpha^2}{2N^2} \quad , \quad N=n+\ell+1 \quad (n,\ell=0,1,2,\ldots) \tag{4.68}$$

that follows from the time-independent Schrödinger equation

$$\Big(-\frac{1}{2m}\Delta+V(r)\Big)\psi_E=E\psi_E \quad \to \quad \Big[-\frac{1}{2m}\Big(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\Big)+\frac{\ell(\ell+1)}{2mr^2}-\frac{Z\alpha}{r}-E_N^{(NR)}\Big]\psi_{N,\ell}=0. \tag{4.69}$$

We will break the calculation into several steps:

1. Denote by $\hat{M}$ the operator that multiplies the factor $\frac{1}{r^2}$ in Eq. (4.67),

$$\hat{M}\equiv\hat{L}^2-Z^2\alpha^2+iZ\alpha\frac{\sigma^i x^i}{r}. \tag{4.70}$$

Since the total angular momentum operator $\hat{J}^i=\hat{L}^i+\frac{1}{2}\sigma^i$ commutes with $\sigma^i x^i$,

$$[\sigma^i x^i,\hat{L}^j+\tfrac{1}{2}\sigma^j]=\sigma^i[x^i,\varepsilon_{jk\ell}x^k\hat{p}^\ell]+\frac{1}{2}[\sigma^i,\sigma^j]x^i=i\varepsilon_{jk\ell}\sigma^i x^k\delta_{i\ell}+i\varepsilon_{ijk}\sigma^k x^i=0, \tag{4.71}$$

the same holds for $\hat{M}$: $[\hat{J}^i,\hat{M}]=0$. (We also used the fact that $\frac{1}{r}$ commutes with $\hat{L}^i$ due to (4.60).)

In the subspace where $\hat{J}^2=j(j+1)$ ($j=\frac{1}{2},\frac{3}{2},\ldots$), and $\hat{J}_3=m$ ($-j\le m\le j$), the orbital quantum number ℓ (such that $\hat{L}^2=\ell(\ell+1)$) can take two values $\ell_\pm=j\pm\frac{1}{2}$. We denote the corresponding eigenstates by $\big|j\,m,\pm\frac{1}{2}\big\rangle$, and for fixed j and m aim to cast $\hat{M}$ in this basis as a 2 by 2 matrix M.

To this end we recall that eigenstates of $\hat{L}^2$ (the spherical functions) have parity $(-1)^\ell$, and since the operator $\frac{\sigma^i x^i}{r}$ changes parity (and is Hermitian), we have

$$\frac{\sigma^i x^i}{r}=\begin{pmatrix}0 & z\\ z^* & 0\end{pmatrix} \quad \text{for some} \quad z\in\mathbb{C}. \tag{4.72}$$

Moreover,

$$\Big(\frac{\sigma^i x^i}{r}\Big)^2=\frac{\sigma^i\sigma^j x^i x^j}{r^2}=\frac{x^i x^i}{r^2}=1 \quad \to \quad \frac{\sigma^i x^i}{r}=\begin{pmatrix}0 & e^{i\theta}\\ e^{-i\theta} & 0\end{pmatrix} \tag{4.73}$$

for some phase θ, which will not influence our results. (This phase factor also absorbs the extra minus sign in the equation for the lower component η_E — see Eq. (4.25).)

Hence, since $\ell_\pm(\ell_\pm+1)=(j+\frac{1}{2})^2\pm(j+\frac{1}{2})$, we find

$$\mathsf{M}=\begin{pmatrix}(j+\frac{1}{2})^2-Z^2\alpha^2+j+\frac{1}{2} & iZ\alpha e^{i\theta}\\ iZ\alpha e^{-i\theta} & (j+\frac{1}{2})^2-Z^2\alpha^2-(j+\frac{1}{2})\end{pmatrix}. \tag{4.74}$$

2. For a 2 by 2 matrix M, the characteristic polynomial, and also the eigenvalues, can be cast in terms of the determinant and the trace:

$$(\lambda-\lambda_+)(\lambda-\lambda_-)=\lambda^2-(\mathrm{Tr}\,\mathsf{M})\lambda+\det\mathsf{M}=0 \quad\to\quad \lambda_\pm=\frac{\mathrm{Tr}\,\mathsf{M}}{2}\pm\sqrt{\left(\frac{\mathrm{Tr}\,\mathsf{M}}{2}\right)^2-\det\mathsf{M}}. \tag{4.75}$$

Since

$$\frac{\mathrm{Tr}\,\mathsf{M}}{2}=(j+\tfrac{1}{2})^2-Z^2\alpha^2 \quad,\quad \det\mathsf{M}=\left((j+\tfrac{1}{2})^2-Z^2\alpha^2\right)^2-(j+\tfrac{1}{2})^2+Z^2\alpha^2, \tag{4.76}$$

we find explicitly

$$\lambda_\pm=\kappa_\pm(\kappa_\pm+1),\quad \text{where}\quad \kappa_+=\sqrt{(j+\tfrac{1}{2})^2-Z^2\alpha^2},\quad \kappa_-=\kappa_+-1. \tag{4.77}$$

Expansion in small $Z\alpha$ yields

$$\kappa_+=\left(j+\frac{1}{2}\right)\sqrt{1-\frac{Z^2\alpha^2}{(j+\frac{1}{2})^2}}=j+\frac{1}{2}-\delta_j,\quad \text{where}\quad \delta_j=\frac{Z^2\alpha^2}{2j+1}+\mathcal{O}(Z^4\alpha^4). \tag{4.78}$$

3. In the eigenbasis corresponding to $\lambda_\pm$ equation (4.67) reads

$$\left[-\frac{1}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)+\frac{\kappa_\pm(\kappa_\pm+1)}{2mr^2}-\frac{Z\alpha}{r}\frac{E}{m}-\frac{E^2-m^2}{2m}\right]\varphi_E^{(\pm)}=0. \tag{4.79}$$

Comparison with the Schrödinger equation (4.69) now yields

$$\frac{E^2-m^2}{2m}=-\frac{mZ^2\alpha^2}{2N'^2}\frac{E^2}{m^2}\quad\to\quad E_{N,j}=\frac{m}{\sqrt{1+\frac{Z^2\alpha^2}{N'^2}}},\quad \text{where}\quad N'=n+\kappa_\pm+1=N-\delta_j, \tag{4.80}$$

and $N=n+\ell_\pm+1$ is a positive integer. (We have neglected the negative part of the energy spectrum.)

4. Finally, we perform an expansion in $(Z\alpha)^2$ with a help of the formulas

$$(1+x)^{-1/2}=1-\frac{1}{2}x+\frac{3}{8}x^2+\mathcal{O}(x^3), \tag{4.81}$$

and

$$\frac{1}{N'^2}=\frac{1}{(N-\delta_j)^2}=\frac{1}{N^2}\left(1+\frac{Z^2\alpha^2}{N(j+\frac{1}{2})}+\mathcal{O}(Z^4\alpha^4)\right). \tag{4.82}$$

The resulting energy levels read

$$\begin{aligned}E_{N,j}&=m\left(1-\frac{Z^2\alpha^2}{2N'^2}+\frac{3}{8}\frac{Z^4\alpha^4}{N'^4}+\mathcal{O}(Z^6\alpha^6)\right)\\&=m\left(1-\frac{Z^2\alpha^2}{2N^2}-\frac{Z^4\alpha^4}{2N^3(j+\frac{1}{2})}+\frac{3}{8}\frac{Z^4\alpha^4}{N^4}+\mathcal{O}(Z^6\alpha^6)\right)\\&=m\underbrace{-\frac{mZ^2\alpha^2}{2N^2}}_{E_N^{(NR)}}\left(1+\frac{Z^2\alpha^2}{N(j+\frac{1}{2})}-\frac{3}{4}\frac{Z^2\alpha^2}{N^2}+\mathcal{O}(Z^4\alpha^4)\right).\end{aligned} \tag{4.83}$$

Exercise 18. *Charge conjugation of a helicity eigenstate.* Consider a positive-energy plane-wave solution of the Dirac equation,

$$\Psi(x) = u(\boldsymbol{p})e^{-ip\cdot x}, \quad \text{where} \quad u(\boldsymbol{p}) = \frac{\sqrt{p_0+m}}{\sqrt{2m}} \begin{pmatrix} \chi \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p_0+m}\chi \end{pmatrix}. \tag{4.84}$$

1. Determine the spin state χ so that the helicity of Ψ is $+\frac{1}{2}$.
2. Find the charge-conjugated wave-function $\Psi_C(x)$.
3. What are energy, momentum, and helicity of $\Psi_C(x)$?

Solution:

1. Application of the helicity operator $\hat{h} = \frac{\boldsymbol{\Sigma}}{2}\cdot\frac{\hat{\boldsymbol{p}}}{|\hat{\boldsymbol{p}}|}$ on Ψ yields

$$\hat{h}\Psi = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{n} & \mathbb{O} \\ \mathbb{O} & \boldsymbol{\sigma}\cdot\boldsymbol{n} \end{pmatrix}\Psi = \frac{1}{2}\frac{\sqrt{p_0+m}}{\sqrt{2m}} \begin{pmatrix} (\boldsymbol{\sigma}\cdot\boldsymbol{n})\chi \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p_0+m}(\boldsymbol{\sigma}\cdot\boldsymbol{n})\chi \end{pmatrix} e^{-ip\cdot x}, \quad \text{where} \quad \boldsymbol{n} \equiv \frac{\hat{\boldsymbol{p}}}{|\hat{\boldsymbol{p}}|}. \tag{4.85}$$

We are therefore looking for $\chi_{\boldsymbol{p}+} \in \mathbb{C}^2$ that satisfies $(\boldsymbol{\sigma}\cdot\boldsymbol{n})\chi_{\boldsymbol{p}+} = \chi_{\boldsymbol{p}+}$. Since $(\boldsymbol{\sigma}\cdot\boldsymbol{n})^2 = |\boldsymbol{n}|^2 = 1$, the matrices

$$\frac{1+\boldsymbol{\sigma}\cdot\boldsymbol{n}}{2} \quad \text{and} \quad \frac{1-\boldsymbol{\sigma}\cdot\boldsymbol{n}}{2} \tag{4.86}$$

are projectors on the eigenspaces of $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ corresponding to the eigenvalues $+1$ and -1, respectively. Projecting an arbitrary reference spinor (for instance $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$) onto the $+1$ eigenspace, and fixing the normalization so that $\chi^\dagger_{\boldsymbol{p}+}\chi_{\boldsymbol{p}+} = 1$, we obtain

$$\chi_{\boldsymbol{p}+} \propto \frac{1+\boldsymbol{\sigma}\cdot\boldsymbol{n}}{2}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+n^3\\ n^1+in^2\end{pmatrix} \quad \rightarrow \quad \chi_{\boldsymbol{p}+} = \frac{1}{\sqrt{2(1+n^3)}}\begin{pmatrix}1+n^3\\ n^1+in^2\end{pmatrix}. \tag{4.87}$$

(Note that this formula is singular for $n^3 = -1$, so in that case a different choice of the reference spinor, for instance $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, has to be made.)

2. The charge conjugation acts on Ψ as $\Psi_C(x) = \gamma^2\Psi^*(x)$. Explicitly,

$$\Psi_C(x) = \frac{\sqrt{p_0+m}}{\sqrt{2m}}\begin{pmatrix}\mathbb{O} & \sigma^2 \\ -\sigma^2 & \mathbb{O}\end{pmatrix}\begin{pmatrix}\chi^*_{\boldsymbol{p}+} \\ \frac{\boldsymbol{\sigma}^*\cdot\boldsymbol{p}}{p_0+m}\chi^*_{\boldsymbol{p}+}\end{pmatrix}e^{ip\cdot x} = \frac{\sqrt{p_0+m}}{\sqrt{2m}}\begin{pmatrix}-\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p_0+m}\sigma^2\chi^*_{\boldsymbol{p}+} \\ -\sigma^2\chi^*_{\boldsymbol{p}+}\end{pmatrix}e^{ip\cdot x}, \tag{4.88}$$

where we have made use of the fact that

$$\sigma^2(\boldsymbol{\sigma}^*\cdot\boldsymbol{p}) = \sigma^2(\sigma^1p^1 - \sigma^2p^2 + \sigma^3p^3) = -(\boldsymbol{\sigma}\cdot\boldsymbol{p})\sigma^2. \tag{4.89}$$

By the same token, we find

$$\sigma^2\chi^*_{\boldsymbol{p}+} = \sigma^2\frac{1+\boldsymbol{\sigma}^*\cdot\boldsymbol{n}}{\sqrt{2(1+n^3)}}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1-\boldsymbol{\sigma}\cdot\boldsymbol{n}}{\sqrt{2(1+n^3)}}\sigma^2\begin{pmatrix}1\\0\end{pmatrix} = \frac{1-\boldsymbol{\sigma}\cdot\boldsymbol{n}}{\sqrt{2(1+n^3)}}\begin{pmatrix}0\\i\end{pmatrix} = i\chi_{\boldsymbol{p}-}. \tag{4.90}$$

In total we have

$$\Psi_C(x) = (-i)\frac{\sqrt{p_0+m}}{\sqrt{2m}}\begin{pmatrix}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p_0+m}\chi_{\boldsymbol{p}-} \\ \chi_{\boldsymbol{p}-}\end{pmatrix}e^{ip\cdot x}, \tag{4.91}$$

which (up to an inessential phase factor $-i$) has the form of a negative-energy plane-wave solution.

3. To determine the energy and momentum of the charge-conjugated wave-function we calculate

$$\hat{p}_0 \Psi_C = i\partial_t \Psi_C = -p_0 \Psi_C \quad \text{and} \quad \hat{p}_i \Psi_C = i\partial_i \Psi_C = -p_i \Psi_C, \tag{4.92}$$

that is, Ψ_C has indeed negative energy and momentum $-\boldsymbol{p}$. The helicity reads

$$\hat{h}\Psi_C = \frac{\boldsymbol{\Sigma}}{2} \cdot \frac{(-\boldsymbol{p})}{|\boldsymbol{p}|} \Psi_C = \frac{1}{2}\Psi_C. \tag{4.93}$$

Charge conjugation flips both the momentum and the spin state of the particle. The helicity is therefore preserved.

Part II

Introduction to Quantum Field Theory

Chapter 5

Classical field theory

5.1 Systems of coupled oscillators

5.1.1 One-dimensional harmonic oscillator

A one-dimensional harmonic oscillator with (angular) frequency ω and mass M is described by the following Lagrangian or Hamiltonian:

$$L(q,\dot{q}) = \frac{M}{2}\dot{q}^2 - \frac{1}{2}M\omega^2 q^2 \quad \rightarrow \quad p = M\dot{q} \quad , \quad H(q,p) = \frac{p^2}{2M} + \frac{1}{2}M\omega^2 q^2. \tag{5.1}$$

The Hamiltonian can be factorised as

$$H(z,z^*) = \omega z^* z, \quad \text{where} \quad z = \sqrt{\frac{M\omega}{2}}q + \frac{ip}{\sqrt{2M\omega}}, \tag{5.2}$$

and the canonical Poisson brackets $\{q,p\}_{PB} = 1$ and $\{q,q\}_{PB} = \{p,p\}_{PB} = 0$ imply

$$\{z,z^*\}_{PB} = -i \quad \text{and} \quad \{z,z\}_{PB} = \{z^*,z^*\}_{PB} = 0. \tag{5.3}$$

(The subscript PB has been added to avoid confusion with an anti-commutator.) The new variables z and z^* are complex conjugate of one another, but they are to be treated as independent linear combinations of the original variables q and p. Canonical equations of motion in the new variables follow easily using the Leibniz rule for Poisson brackets,

$$\dot{z} = \{z,H\}_{PB} = \omega z^*\{z,z\}_{PB} + \omega\{z,z^*\}_{PB} z = -i\omega z, \tag{5.4}$$

with the solution for arbitrary initial condition $z(0) = A \in \mathbb{C}$ given by

$$z(t) = A\,e^{-i\omega t} \quad \rightarrow \quad q(t) = \sqrt{\frac{2}{M\omega}}\,\mathrm{Re}\,z(t) = \frac{z(t)+z^*(t)}{\sqrt{2M\omega}} = \frac{1}{\sqrt{2M\omega}}\Big(A\,e^{-i\omega t} + A^* e^{i\omega t}\Big). \tag{5.5}$$

5.1.2 Normal modes

Let us now consider a system of N coupled degrees of freedom q_n of equal masses M described by the quadratic Lagrangian

$$L(q_n,\dot{q}_n) = \sum_n \frac{M}{2}\dot{q}_n^2 - \frac{1}{2}\sum_{n,n'} U_{nn'} q_n q_{n'} = \frac{M}{2}\dot{\vec{q}}^T\dot{\vec{q}} - \frac{1}{2}\vec{q}^T \mathsf{U}\,\vec{q}, \tag{5.6}$$

where $\vec{q} = (q_n) \in \mathbb{R}^N$ is the vector of displacements from equilibrium positions, and $\mathsf{U} = (U_{nn'})$ is an $N \times N$ real symmetric positive-definite matrix — the 'potential energy matrix'. In the Hamiltonian formulation we have

$$p_n = \frac{\partial L}{\partial \dot{q}_n} = M\dot{q}_n \quad \rightarrow \quad H(q_n, p_n) = \sum_n \frac{p_n^2}{2M} + \frac{1}{2}\sum_{n,n'} U_{nn'} q_n q_{n'}. \tag{5.7}$$

From the treatment of small oscillations of classical systems [9, Ch. 1.7] we know that it is convenient to diagonalise the matrix U,

$$\mathsf{U} = \mathsf{V}\, M\Omega^2\, \mathsf{V}^T, \quad \text{where} \quad \Omega = \mathrm{diag}(\omega_1, \ldots, \omega_N), \quad \omega_1 \leq \ldots \leq \omega_N, \quad \text{and} \quad \mathsf{V}^T\mathsf{V} = \mathsf{V}\mathsf{V}^T = \mathbb{I}, \tag{5.8}$$

and perform the transformation to normal coordinates $\vec{\eta} = (\eta_k)$, i.e., express

$$\vec{q}(t) = \mathsf{V}\,\vec{\eta}(t) = \sum_{k=1}^{N} \vec{v}_k \eta_k(t), \quad \text{where} \quad \mathsf{V} = (\vec{v}_1, \ldots, \vec{v}_N). \tag{5.9}$$

Here the vectors $\vec{v}_k$ form an orthonormal basis of the configuration space $\mathbb{R}^N$:

$$\begin{aligned} \text{orthonormality :} \quad & (\mathsf{V}^T\mathsf{V})_{kk'} = \vec{v}_k^T \vec{v}_{k'} = \delta_{kk'} \\ \text{completeness :} \quad & \mathsf{V}\mathsf{V}^T = \sum_k \vec{v}_k \vec{v}_k^T = \mathbb{I}. \end{aligned} \tag{5.10}$$

They are the *normal modes* of the oscillating system (collective degrees of freedom).

The Lagrangian, when expressed in the normal coordinates, gets decoupled,

$$L(\eta_k, \dot{\eta}_k) = \frac{M}{2}\dot{\vec{\eta}}^T \mathsf{V}^T\mathsf{V}\dot{\vec{\eta}} - \frac{1}{2}\vec{\eta}^T\mathsf{V}^T\mathsf{U}\mathsf{V}\,\vec{\eta} = \sum_k \left(\frac{M}{2}\dot{\eta}_k^2 - \frac{M}{2}\omega_k^2\eta_k^2\right) \quad \rightarrow \quad \ddot{\eta}_k + \omega_k^2\eta_k = 0 \quad (\forall k). \tag{5.11}$$

It describes a system of N independent harmonic oscillators with coordinates η_k, and frequencies ω_k.

In the Hamiltonian formulation, the canonical momenta corresponding to the normal coordinates read $\rho_k = M\dot{\eta}_k$, and the Hamiltonian is

$$H(\eta_k, \rho_k) = \sum_k \left(\frac{\rho_k^2}{2M} + \frac{M}{2}\omega_k^2\eta_k^2\right) \quad \rightarrow \quad H(z_k, z_k^*) = \sum_k \omega_k z_k^* z_k, \quad z_k = \sqrt{\frac{M\omega_k}{2}}\eta_k + \frac{i\rho_k}{\sqrt{2M\omega_k}}. \tag{5.12}$$

The Poisson brackets between the z-variables, and the ensuing equations of motion read

$$\{z_k, z_{k'}^*\}_{PB} = -i\delta_{kk'}, \quad \{z_k, z_{k'}\}_{PB} = \{z_k^*, z_{k'}^*\}_{PB} = 0 \quad \rightarrow \quad \dot{z}_k = \{z_k, H\}_{PB} = -i\omega_k z_k. \tag{5.13}$$

Equation (5.5) then readily provides the trajectories $z_k(t)$ and also $\eta_k(t)$, which can be superposed via Eq. (5.9) to yield a general solution for the oscillating system

$$\vec{q}(t) = \sum_k \frac{\vec{v}_k}{\sqrt{2M\omega_k}}\Big(A_k e^{-i\omega_k t} + A_k^* e^{i\omega_k t}\Big). \tag{5.14}$$

Although the individual masses (the degrees of freedom q_n) influence each other as they are coupled by springs, the normal modes, which we have been able to find owing to the quadratic nature of Lagrangian (5.6), are independent — their coordinates η_k evolve independently of each other.

5.1.3 Linear chain and string

Let us assume that the masses M are arranged in a linear chain, and connected by springs characterized by a spring constant κ. The Lagrangian then reads

$$L(q_n, \dot{q}_n) = \sum_n \frac{M}{2}\dot{q}_n^2 - \sum_n \frac{\kappa}{2}(q_{n+1} - q_n)^2. \tag{5.15}$$

(We envisage an infinite chain without specific boundary conditions.)

We wish to pass to the continuum limit, where the equilibrium interparticle distance $a \to 0$. Introducing a function $\phi(x,t)$, $x \in \mathbb{R}$, we may write

$$\phi(na,t) = q_n(t) \quad \to \quad L = \sum_n a \frac{M}{2a}\dot{\phi}(na)^2 - \sum_n a \frac{\kappa a}{2}\left(\frac{\phi(na+a) - \phi(na)}{a}\right)^2. \tag{5.16}$$

Identifying $a = dx$, $na = x$, $\frac{M}{a} = \rho$ (linear density), and $\kappa a = T$ (string tension), the Lagrangian becomes

$$L = \int dx\, \mathscr{L}, \quad \text{with} \quad \mathscr{L} = \frac{\rho}{2}(\partial_t \phi)^2 - \frac{T}{2}(\partial_x \phi)^2. \tag{5.17}$$

In the continuum limit, the spring coupling is captured by the spatial derivative term.

The corresponding field-theoretical Euler-Lagrange equation of motion

$$\partial_t \frac{\partial \mathscr{L}}{\partial(\partial_t \phi)} + \partial_x \frac{\partial \mathscr{L}}{\partial(\partial_x \phi)} - \frac{\partial \mathscr{L}}{\partial \phi} = \rho \frac{\partial^2 \phi}{\partial t^2} - T \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{5.18}$$

is the wave equation of a one-dimensional (infinite) string. This can be further rewritten as

$$\left(\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi(x,t) = 0, \quad \text{where} \quad c_s \equiv \sqrt{\frac{T}{\rho}} \tag{5.19}$$

is the speed of sound, i.e., the speed at which waves on the string propagate.

It is interesting to note that the wave equation is invariant under "Lorentz" boosts, with c_s playing the role of the speed of light in special relativity. We arrive at a "relativistic" equation starting from a non-relativistic mechanical system of oscillating masses.

5.2 Functional derivatives

Functionals generalize the notion of functions of several variables to the case of (continuously) infinitely many independent variables:

$$f : u = (u_1, \ldots, u_n) \mapsto f(u) \quad \rightsquigarrow \quad F : \phi \mapsto F[\phi]. \tag{5.20}$$

The infinitely many degrees of freedom of the functional F are collected in a function $\phi(x)$, defined on a certain domain in $\mathbb{R}^D$, and satisfying certain smoothness and boundary conditions.

Let us write the increment of a function f when moving from a point u in the direction $h = (h_1, \ldots, h_n)$, and analogously the increment of a functional F when varying ϕ in the "direction" in the function space specified by a function η:

$$f(u + \varepsilon h) - f(u) \approx \varepsilon \sum_i \frac{\partial f(u)}{\partial u_i} h_i \quad \rightsquigarrow \quad F[\phi + \varepsilon \eta] - F[\phi] \approx \varepsilon \int dx\, \frac{\delta F[\phi]}{\delta \phi(x)} \eta(x). \tag{5.21}$$

In more technical terms, we are using the Riesz lemma to express the derivative (which is at each point a linear functional of the direction) as a scalar product of the direction with the gradient $\frac{\partial f}{\partial u_i}$, or the "functional gradient" $\frac{\delta F}{\delta \phi(x)}$, respectively.

Without further aspiration at mathematical rigour, we will introduce functional (or variational) derivatives by analogy with partial derivatives of multivariate functions. Partial derivative with respect to the jth independent variable is obtained by setting $h_i = \delta_{ij}$; similarly, the *functional derivative* at point y is obtained by setting $\eta(x) = \delta(x-y)$:

$$\frac{\partial f(u)}{\partial u_j} = \lim_{\varepsilon\to 0} \frac{f(u_1,\ldots,u_j+\varepsilon,\ldots,u_n) - f(u)}{\varepsilon} \quad \rightsquigarrow \quad \frac{\delta F[\phi]}{\delta\phi(y)} = \lim_{\varepsilon\to 0} \frac{F[\phi(x)+\varepsilon\delta(x-y)] - F[\phi(x)]}{\varepsilon}. \tag{5.22}$$

In this expression x is a 'silent' variable that enumerates the degrees of freedom of a function ϕ, much like an index i can be used to enumerate the components u_i of u. The Dirac δ-function that features in the argument of F can be understood as a nascent δ-function, i.e., a regularizing sequence of ordinary functions.

As a basic example, let us consider, for a fixed x, the functional $F_x[\phi] = \phi(x)$, and calculate

$$\frac{\delta F_x[\phi]}{\delta\phi(y)} = \lim_{\varepsilon\to 0} \frac{\phi(x)+\varepsilon\delta(x-y) - \phi(x)}{\varepsilon} = \delta(x-y), \qquad \text{that is} \qquad \frac{\delta\phi(x)}{\delta\phi(y)} = \delta(x-y) \tag{5.23}$$

(which is to be compared with $\frac{\partial u_i}{\partial u_j} = \delta_{ij}$). Having this simple formula in mind in fact allows one to calculate plenty of results, since functional derivatives exhibit a number of properties, which are unsurprisingly analogous to the properties of partial derivatives.

1. Linearity:

$$\frac{\delta}{\delta\phi(y)}\big(\alpha(x)F[\phi] + G[\phi]\big) = \alpha(x)\frac{\delta F[\phi]}{\delta\phi(y)} + \frac{\delta G[\phi]}{\delta\phi(y)}. \tag{5.24}$$

   For example, we can easily calculate

$$\frac{\delta}{\delta\phi(y)}\int dx\, \phi(x)\psi(x) = \int dx\, \delta(x-y)\psi(x) = \psi(y) \quad , \quad \frac{\delta}{\delta\phi(y)}\partial_x\phi(x) = \partial_x\delta(x-y). \tag{5.25}$$

2. Product (or Leibniz) rule:

$$\frac{\delta}{\delta\phi(y)}\big(F[\phi]G[\phi]\big) = \frac{\delta F[\phi]}{\delta\phi(y)}G[\phi] + F[\phi]\frac{\delta G[\phi]}{\delta\phi(y)}. \tag{5.26}$$

3. There are various ways to compose functions and functionals and hence various chain rules. We mention

$$\frac{\delta}{\delta\phi(y)}g(F[\phi]) = g'(F[\phi])\frac{\delta F[\phi]}{\delta\phi(y)} \tag{5.27}$$

   (provided F takes values in the domain of the function g).

   Then, for example,

$$\frac{\delta}{\delta\phi(y)}\int dx\, g(\phi(x)) = \int dx\, g'(\phi(x))\delta(x-y) = g'(\phi(y)). \tag{5.28}$$

   An important generalisation of this result to integral functionals that depend also on derivatives of ϕ, e.g., action functionals, is provided in Exercise 20.

4. Functional Taylor series is analogous to the Taylor series of a multivariate function:

$$f(u+h) = f(u) + \sum_i \frac{\partial f(u)}{\partial u_i} h_i + \frac{1}{2!}\sum_{ij} \frac{\partial^2 f(u)}{\partial u_i \partial u_j} h_i h_j + \dots$$
$$\rightsquigarrow \quad F[\phi+\eta] = F[\phi] + \int dx\, \frac{\delta F[\phi]}{\delta\phi(x)}\,\eta(x) + \frac{1}{2!}\int dxdy\, \frac{\delta^2 F[\phi]}{\delta\phi(x)\delta\phi(y)}\,\eta(x)\eta(y) + \dots. \tag{5.29}$$

5.3 Lagrangian and Hamiltonian formalism in field theory

We move from analytical mechanics of (possibly many) discrete degrees of freedom $q_n(t)$, where $n = 1, \dots, N$, to a field theory of (continuously) infinitely many degrees of freedom $\phi(\boldsymbol{x}, t)$, where $\boldsymbol{x}$ are points in $\mathbb{R}^3$. In this section we will consider, for simplicity, a one-component field ϕ, i.e., one degree of freedom per spatial point $\boldsymbol{x}$; generalisation to several components ϕ_r (forming the so-called 'internal space' of the theory) is straightforward when needed.

In Lagrangian formalism, the Lagrangian of a discrete system is replaced by a field Lagrangian,

$$L(q_n, \dot{q}_n) \quad \rightsquigarrow \quad L[\phi(\boldsymbol{x}), \dot{\phi}(\boldsymbol{x})], \tag{5.30}$$

which is a functional of two fields: the 'displacement' field $\phi(\boldsymbol{x})$, and the 'velocity' field $\dot{\phi}(\boldsymbol{x})$. Time evolution adds time dependence of the fields, $\phi(\boldsymbol{x}, t)$, $\dot{\phi}(\boldsymbol{x}, t)$, fixing the velocity field as the time derivative of the displacement field. Dynamics is governed by the action,

$$S[q_n(t)] = \int_{t_1}^{t_2} dt\, L(q_n(t), \dot{q}_n(t)) \quad \rightsquigarrow \quad S[\phi(\boldsymbol{x}, t)] = \int_{t_1}^{t_2} dt\, L[\phi(\boldsymbol{x}, t), \dot{\phi}(\boldsymbol{x}, t)], \tag{5.31}$$

with the ensuing Euler-Lagrange equations of motion

$$\frac{\delta S}{\delta q_n(t)} = \frac{\partial L}{\partial q_n} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} = 0 \quad \rightsquigarrow \quad \frac{\delta S}{\delta\phi(\boldsymbol{x}, t)} = \frac{\delta L}{\delta\phi(\boldsymbol{x})} - \frac{d}{dt}\frac{\delta L}{\delta\dot{\phi}(\boldsymbol{x})} = 0. \tag{5.32}$$

For local field theories, the Lagrangian and the action assume the form

$$L[\phi(\boldsymbol{x}), \dot{\phi}(\boldsymbol{x})] = \int d^3x\, \mathscr{L}(\phi(\boldsymbol{x}), \nabla\phi(\boldsymbol{x}), \dot{\phi}(\boldsymbol{x})) \quad , \quad S[\phi(x)] = \int d^4x\, \mathscr{L}(\phi(x), \partial_\mu\phi(x)), \tag{5.33}$$

where the function $\mathscr{L}$ is referred to as the *Lagrangian density.* The Lagrangian L is therefore a "continuous sum" of terms, each one depending of the value of ϕ at a given point $\boldsymbol{x}$ and its infinitesimal neighbourhood (via the derivatives).

Apart from Section 6.2.3 (and Exercise 25), where we briefly discuss the field-theoretic description of non-relativistic interacting many-particle systems, we will deal exclusively with local field theories (and often use the term 'Lagrangian' in place of 'Lagrangian density' for brevity). Then, by Exercise 20 with $D = 3$, the equations of motion (5.32) become

$$\frac{\partial\mathscr{L}}{\partial\phi} - \partial_i\frac{\partial\mathscr{L}}{\partial(\partial_i\phi)} - \frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{\phi}} = \frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} = 0. \tag{5.34}$$

The same result can be obtained directly by equating to zero the functional derivative $\frac{\delta S}{\delta\phi(x)}$ of the action in Eq. (5.33) with a help of Exercise 20 (the case $D = 4$). For later reference we quote the resulting field-theoretic Euler-Lagrange equations also for a multicomponent field (ϕ_r):

$$\frac{\partial\mathscr{L}}{\partial\phi_r} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)} = 0 \qquad (\forall r). \tag{5.35}$$

To provide some examples, we list Lagrangian densities that lead to the Klein-Gordon (see Exercise 21), the Dirac (Exercise 22), and the Maxwell equations (Eq. (10.2)), respectively:

$$\begin{aligned} \text{Klein-Gordon field:} \quad & \mathscr{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2, \\ \text{Dirac field:} \quad & \mathscr{L}(\Psi, \bar{\Psi}, \partial_\mu \Psi, \partial_\mu \bar{\Psi}) = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi, \\ \text{Electromagnetic field:} \quad & \mathscr{L}(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{5.36}$$

The passage from the Lagrangian to the Hamiltonian formalism starts by defining the canonical momenta p_n associated with the degrees of freedom q_n, or, in continuous field theory, the momentum field $\pi(\boldsymbol{x})$ associated with the displacement field $\phi(\boldsymbol{x})$:

$$p_n = \frac{\partial L}{\partial \dot{q}_n}(q_n, \dot{q}_n) \quad \rightsquigarrow \quad \pi(\boldsymbol{x}) = \frac{\delta L}{\delta \dot{\phi}(\boldsymbol{x})}[\phi(\boldsymbol{x}), \dot{\phi}(\boldsymbol{x})] = \frac{\partial \mathscr{L}}{\partial \dot{\phi}}(\phi(\boldsymbol{x}), \nabla\phi(\boldsymbol{x}), \dot{\phi}(\boldsymbol{x})). \tag{5.37}$$

For a discrete system the Hamiltonian reads

$$H(q_n, p_n) = \sum_n p_n \dot{q}_n - L(q_n, \dot{q}_n), \quad \text{where} \quad \dot{q}_n = \dot{q}_n(q_m, p_m); \tag{5.38}$$

for a field system, analogously,

$$H[\phi(\boldsymbol{x}), \pi(\boldsymbol{x})] = \int d^3x \left(\pi(\boldsymbol{x})\dot{\phi}(\boldsymbol{x}) - \mathscr{L}\right) = \int d^3x\, \mathscr{H}, \tag{5.39}$$

where we have defined the Hamiltonian density

$$\mathscr{H}(\phi, \nabla\phi, \pi) = \pi\, \dot{\phi} - \mathscr{L}(\phi, \nabla\phi, \dot{\phi}) = \frac{\partial \mathscr{L}}{\partial(\partial_0 \phi)}\, \partial_0 \phi - \mathscr{L}. \tag{5.40}$$

(We will later identify this quantity as the component $\mu = \nu = 0$ of the canonical energy momentum tensor $T^\mu{}_\nu$ — see Eq. (5.60).)

The field-theoretic Poisson bracket of two functionals $F[\phi, \pi]$ and $G[\phi, \pi]$ is defined as

$$\{F, G\}_{PB} = \int d^3x \left(\frac{\delta F}{\delta\phi(\boldsymbol{x})}\frac{\delta G}{\delta\pi(\boldsymbol{x})} - \frac{\delta G}{\delta\phi(\boldsymbol{x})}\frac{\delta F}{\delta\pi(\boldsymbol{x})}\right) = -\{G, F\}_{PB}. \tag{5.41}$$

For simple functionals of the form $\phi(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$ we find, in particular, the canonical Poisson brackets

$$\{\phi(\boldsymbol{x}), \pi(\boldsymbol{y})\}_{PB} = \int d^3x' \left(\frac{\delta\phi(\boldsymbol{x})}{\delta\phi(\boldsymbol{x}')}\frac{\delta\pi(\boldsymbol{y})}{\delta\pi(\boldsymbol{x}')} - \frac{\delta\pi(\boldsymbol{y})}{\delta\phi(\boldsymbol{x}')}\frac{\delta\phi(\boldsymbol{x})}{\delta\pi(\boldsymbol{x}')}\right) = \int d^3x'\, \delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\boldsymbol{y}-\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{y}), \tag{5.42}$$

and

$$\{\phi(\boldsymbol{x}), \phi(\boldsymbol{y})\}_{PB} = \{\pi(\boldsymbol{x}), \pi(\boldsymbol{y})\}_{PB} = 0, \tag{5.43}$$

which is a continuum version of the canonical relations $\{q_n, p_{n'}\}_{PB} = \delta_{nn'}$, and $\{q_n, q_{n'}\}_{PB} = \{p_n, p_{n'}\}_{PB} = 0$.

Finally, let us observe that Hamilton's canonical equations of motion for the fields $\phi(\boldsymbol{x}, t)$ and $\pi(\boldsymbol{x}, t)$ can be expressed with a help of Poisson brackets as (all fields taken at point $\boldsymbol{x}$ and time t)

$$\begin{aligned} \partial_t \phi &= \{\phi, H\}_{PB} = \int d^3x' \frac{\delta\phi(\boldsymbol{x})}{\delta\phi(\boldsymbol{x}')}\frac{\delta H}{\delta\pi(\boldsymbol{x}')} = \frac{\delta H}{\delta\pi(\boldsymbol{x})} = \frac{\partial\mathscr{H}}{\partial\pi}, \\ \partial_t \pi &= \{\pi, H\}_{PB} = -\int d^3x' \frac{\delta H}{\delta\phi(\boldsymbol{x}')}\frac{\delta\pi(\boldsymbol{x})}{\delta\pi(\boldsymbol{x}')} = -\frac{\delta H}{\delta\phi(\boldsymbol{x})} = -\frac{\partial\mathscr{H}}{\partial\phi} + \partial_i \frac{\partial\mathscr{H}}{\partial(\partial_i\phi)}. \end{aligned} \tag{5.44}$$

5.4 Symmetries and conservation laws

Noether's theorem provides a link between symmetry properties of a physical theory and conservation laws. Here a 'symmetry' explicitly means an invariance of the action under certain transformation of spacetime coordinates and fields, and conservation laws take the form of continuity equations for the respective (conserved) four-currents.

Let us consider the variations

$$x'^{\mu} = x^{\mu} + \delta x^{\mu}(x) \quad , \quad \phi'_r(x') = \phi_r(x) + \delta\phi_r(x), \tag{5.45}$$

and investigate the change of the action of a local field theory with Lagrangian density $\mathscr{L}$,

$$\delta S \equiv \int_{\Omega'} d^4x'\, \mathscr{L}(\phi'_r(x'), \partial'_\mu \phi'_r(x'), x') - \int_{\Omega} d^4x\, \mathscr{L}(\phi_r(x), \partial_\mu \phi_r(x), x), \tag{5.46}$$

where Ω is an arbitrary spacetime region. (All equations in this section will be understood as valid up to first order in the variations δx^μ and $\delta\phi_r$.)

We start with the transformation of the four-volume element:

$$d^4x' = \left|\det\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)\right| d^4x = \det\left(\delta^\mu_\nu + \partial_\nu \delta x^\mu\right) d^4x = (1+\partial_0\delta x^0)\cdots(1+\partial_3\delta x^3)d^4x = (1+\partial_\mu\delta x^\mu)d^4x. \tag{5.47}$$

Furthermore, to carry out the substitution of variables $x'(x)$ in the primed action functional we calculate

$$\partial_\mu \phi'_r(x'(x)) = \frac{\partial x'^{\nu}}{\partial x^{\mu}} \partial'_\nu \phi'_r(x') = (\delta^\nu_\mu + \partial_\mu \delta x^\nu)\partial'_\nu \phi'_r(x') = \partial'_\mu \phi'_r(x') + (\partial_\mu \delta x^\nu)\partial_\nu \phi_r(x), \tag{5.48}$$

where in the very last term we could drop the primes as this makes a difference of the second order, and express

$$\partial'_\mu \phi'_r(x')\big|_{x'(x)} = \partial_\mu \phi_r(x) + \partial_\mu \delta\phi_r(x) - (\partial_\mu \delta x^\nu)\partial_\nu \phi_r(x). \tag{5.49}$$

Now we perform the substitution in the integral (writing $\mathscr{L} \equiv \mathscr{L}(\phi_r(x), \partial_\mu\phi_r(x), x)$):

$$\begin{aligned}
\delta S &= \int_\Omega d^4x \left[(1+\partial_\mu\delta x^\mu)\mathscr{L}\big(\phi_r + \delta\phi_r, \partial_\mu\phi_r + \partial_\mu\delta\phi_r - (\partial_\mu\delta x^\nu)\partial_\nu\phi_r, x+\delta x\big) - \mathscr{L}\right] \\
&= \int_\Omega d^4x \left[\frac{\partial\mathscr{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}\big(\partial_\mu\delta\phi_r - (\partial_\mu\delta x^\nu)\partial_\nu\phi_r\big) + \left.\frac{\partial\mathscr{L}}{\partial x^\mu}\right|_{expl}\delta x^\mu + (\partial_\mu\delta x^\mu)\mathscr{L}\right].
\end{aligned} \tag{5.50}$$

(Here the subscript "*expl*" denotes explicit derivative of the Lagrangian $\mathscr{L}$.) Next, we rewrite the last term as $(\partial_\mu\delta x^\mu)\mathscr{L} = \partial_\mu(\delta x^\mu\mathscr{L}) - \delta x^\mu\partial_\mu\mathscr{L}$, and use the expression for the total derivative of the Lagrangian,

$$\delta x^\mu\partial_\mu\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\phi_r}\delta x^\mu\partial_\mu\phi_r + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}\delta x^\nu\partial_\nu\partial_\mu\phi_r + \left.\frac{\partial\mathscr{L}}{\partial x^\mu}\right|_{expl}\delta x^\mu, \tag{5.51}$$

to obtain

$$\begin{aligned}
\delta S &= \int_\Omega d^4x \left[\frac{\partial\mathscr{L}}{\partial\phi_r}(\delta\phi_r - \delta x^\mu\partial_\mu\phi_r) + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}\big(\partial_\mu\delta\phi_r - (\partial_\mu\delta x^\nu)\partial_\nu\phi_r - \delta x^\nu\partial_\nu\partial_\mu\phi_r\big) + \partial_\mu(\delta x^\mu\mathscr{L})\right] \\
&= \int_\Omega d^4x \left[\frac{\partial\mathscr{L}}{\partial\phi_r}\bar{\delta}\phi_r + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}\partial_\mu\bar{\delta}\phi_r + \partial_\mu(\delta x^\mu\mathscr{L})\right] \\
&= \int_\Omega d^4x \left[\left(\frac{\partial\mathscr{L}}{\partial\phi_r} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}\right)\bar{\delta}\phi_r + \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}\bar{\delta}\phi_r + \delta x^\mu\mathscr{L}\right)\right],
\end{aligned} \tag{5.52}$$

where we have denoted $\bar{\delta}\phi_r \equiv \delta\phi_r - \delta x^\nu \partial_\nu \phi_r$.

Now comes the Noether theorem. If we assume that the functions $\phi_r(x)$ are solutions of the Euler-Lagrange equations of motion, the first term vanishes. If, in addition, $\delta S = 0$ for any choice of the spacetime region Ω, i.e., the action functionals in unprimed and primed variables are equal (the transformation is a *symmetry* of the theory with action $\int d^4x\, \mathscr{L}$), then the following local conservation law (or continuity equation) holds:

$$\partial_\mu f^\mu = 0, \tag{5.53}$$

where

$$\begin{aligned} f^\mu &\equiv \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)}(\delta\phi_r - \delta x^\nu \partial_\nu \phi_r) + \delta x^\mu \mathscr{L} \\ &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)}\delta\phi_r - \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)}\partial_\nu \phi_r - \delta^\mu_\nu \mathscr{L}\right)\delta x^\nu \end{aligned} \tag{5.54}$$

is a conserved *Noether current*.

This local conservation law can be integrated over a three-dimensional volume V to yield

$$\partial_t \int_V d^3x\, f^0 = -\int_V d^3x\, \partial_i f^i = -\oint_{\partial V} d\Sigma_i\, f^i, \tag{5.55}$$

where in the last step we have applied the Gauss theorem. If the spatial volume V approaches the whole of $\mathbb{R}^3$, and the fields ϕ_r together with their derivatives fall off sufficiently fast at infinity, then the surface integral on the right vanishes, and the integrated quantity (the *Noether charge*) $\int d^3x f^0(\boldsymbol{x}, t)$ is time-independent.

In the following subsections we will list some common symmetries of field theories, and derive the corresponding conserved quantities. The variations (5.45) get parametrized by a finite set of continuous parameters, each generating one Noether current.

5.4.1 Translations and energy-momentum tensor

Consider the variations

$$x'^\mu = x^\mu + \varepsilon a^\mu \quad , \quad \phi'_r(x') = \phi_r(x), \tag{5.56}$$

where $a = (a^\mu)$ is a constant spacetime vector, and ε is infinitesimally small. Now, since

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu \quad \to \quad d^4x' = d^4x \quad , \quad \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = \frac{\partial}{\partial x'^\mu}, \tag{5.57}$$

the variation of the action, Eq. (5.46), reduces to

$$\delta S = \int_\Omega d^4x \left[\mathscr{L}(\phi_r(x), \partial_\mu\phi_r(x), x + \varepsilon a) - \mathscr{L}(\phi_r(x), \partial_\mu\phi_r(x), x)\right] \approx \varepsilon \int_\Omega d^4x\, a^\mu \left.\frac{\partial \mathscr{L}}{\partial x^\mu}\right|_{expl}. \tag{5.58}$$

Assuming that the Lagrangian does not depend explicitly on x, the transformation (5.56) is a symmetry of the theory ($\delta S = 0$), and we have from Eq. (5.54), with $\delta x^\nu = \varepsilon a^\nu$ and $\delta\phi_r = 0$,

$$f^\mu = -\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)}\partial_\nu \phi_r - \delta^\mu_\nu \mathscr{L}\right)\varepsilon\, a^\nu. \tag{5.59}$$

Since a^μ is arbitrary, we thus obtain (after dividing by ε) four independent conservation laws

$$\partial_\mu T^\mu{}_\nu = 0, \quad \text{where} \quad T^\mu{}_\nu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)}\partial_\nu \phi_r - \delta^\mu_\nu \mathscr{L}. \tag{5.60}$$

The quantity $T^\mu{}_\nu$ is called the *canonical energy-momentum* (or *stress-energy*) *tensor*. The sign difference between f^μ and $T^\mu{}_\nu$ is a matter of convention, ensuring that the component $T^0{}_0$ (the energy density) coincides with the Hamiltonian density $\mathscr{H}$ of Eq. (5.40). The spatial vector field $T^i{}_0$ is then the energy flux. The components

$$T^0{}_j = \frac{\partial \mathscr{L}}{\partial(\partial_0 \phi_r)} \partial_j \phi_r = \pi_r \partial_j \phi_r \tag{5.61}$$

represent the (density of) field momentum, with $T^i{}_j$ being the respective field momentum fluxes. Integrating over $\mathbb{R}^3$ we obtain the *total four-momentum*

$$P_\nu = \int d^3x\, T^0{}_\nu. \tag{5.62}$$

Mind the difference between the canonical momentum π_r, and the field momentum $\pi_r \partial_j \phi_r$. The canonical momentum relates to the velocity of the field at individual points (the velocity of individual masses in a discretized picture), while the field momentum relates to the propagation of waves (it vanishes if ϕ_r are spatially constant).

As an illustration, consider a one-dimensional mechanical example — the string from Section 5.1.3. This has the Lagrangian density

$$\mathscr{L} = \frac{\rho}{2}(\partial_t \phi)^2 - \frac{T}{2}(\partial_x \phi)^2, \qquad \text{which yields} \qquad \frac{\partial \mathscr{L}}{\partial(\partial_t \phi)} = \rho \partial_t \phi \quad , \quad \frac{\partial \mathscr{L}}{\partial(\partial_x \phi)} = -T\partial_x \phi. \tag{5.63}$$

The canonical energy-momentum tensor is a 2 by 2 matrix

$$\begin{pmatrix} T^t{}_t & T^t{}_x \\ T^x{}_t & T^x{}_x \end{pmatrix} = \begin{pmatrix} \frac{\rho}{2}(\partial_t \phi)^2 + \frac{T}{2}(\partial_x \phi)^2 & \rho(\partial_t \phi)(\partial_x \phi) \\ -T(\partial_x \phi)(\partial_t \phi) & -\frac{\rho}{2}(\partial_t \phi)^2 - \frac{T}{2}(\partial_x \phi)^2 \end{pmatrix}, \tag{5.64}$$

where the first column contains the energy density $T^t{}_t$ and energy flux $T^t{}_x$ (introduced in detail in Ref. [9, Ch. 2.9]), and the second column contains the momentum density $T^x{}_t$ and momentum flux $T^x{}_x$.

The canonical energy-momentum tensors corresponding to the three Lagrangians in Eq. (5.36) read:

$$\begin{aligned} \text{Klein-Gordon field:} \quad & T^\mu{}_\nu = \partial^\mu \phi\, \partial_\nu \phi - \delta^\mu_\nu \frac{1}{2}(\partial_\rho \phi\, \partial^\rho \phi - m^2 \phi^2), \\ \text{Dirac field:} \quad & T^\mu{}_\nu = i\bar{\Psi}\gamma^\mu \partial_\nu \Psi - \delta^\mu_\nu \bar{\Psi}(i\gamma^\rho \partial_\rho - m)\Psi, \\ \text{Electromagnetic field:} \quad & T^\mu{}_\nu = -F^{\mu\rho}\partial_\nu A_\rho + \delta^\mu_\nu \frac{1}{4} F_{\rho\sigma}F^{\rho\sigma}. \end{aligned} \tag{5.65}$$

In this list only the Klein-Gordon $T_{\mu\nu}$ is symmetric, and as such can be directly used as a source of gravitational field in Einstein's field equations of general relativity,

$$G_{\mu\nu} = \kappa T_{\mu\nu}. \tag{5.66}$$

Here $\kappa = \frac{8\pi G}{c^4}$ is the Einstein gravitational constant, and the Einstein tensor $G_{\mu\nu}$ is symmetric by construction. The other two canonical energy-momentum tensors can be symmetrized via an algorithm described in [2, p. 47] in a way that does not spoil the conservation law (5.60).

In passing, let us remark that there exists a derivation of the canonical energy-momentum tensor, which is much simpler than the general procedure of Section 5.4. Taking the total derivative of a Lagrangian $\mathscr{L} \equiv \mathscr{L}(\phi_r(x), \partial_\mu \phi_r(x))$ that does not explicitly depend on x, we find

$$\partial_\nu \mathscr{L} = \frac{\partial \mathscr{L}}{\partial \phi_r}\partial_\nu \phi_r + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)}\partial_\nu \partial_\mu \phi_r = \left(\frac{\partial \mathscr{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)}\right)\partial_\nu \phi_r + \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)}\partial_\nu \phi_r\right). \tag{5.67}$$

If we now impose the Euler-Lagrange equations of motion, and write $\partial_\nu \mathscr{L}$ as $\partial_\mu(\delta^\mu_\nu \mathscr{L})$, we obtain

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)} \partial_\nu \phi_r - \delta^\mu_\nu \mathscr{L} \right) = 0. \tag{5.68}$$

5.4.2 Lorentz transformations and angular momentum tensor

Consider Lorentz transformations $\mathsf{L} = \exp\left(-\frac{i}{2}\varepsilon\, \omega_{\mu\nu} \mathsf{M}^{\mu\nu}\right)$ for infinitesimal values of ε. Recalling Eq. (2.19), the corresponding variations are

$$x'^\mu = x^\mu + \varepsilon\, \omega^\mu{}_\nu x^\nu \quad , \quad \phi'_r(x') = \phi_r(x) - \frac{i}{2}\varepsilon\, \omega_{\mu\nu} (\mathsf{S}^{\mu\nu})_{rs} \phi_s(x), \tag{5.69}$$

where the (effectively 6) matrices $\mathsf{S}^{\mu\nu}$ form a certain representation of the Lorentz algebra. For example, $\mathsf{S}^{\mu\nu} = 0$ for a scalar (spin-0) field, which undergoes trivial internal Lorentz transformations; $\mathsf{S}^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}$ for a spinor (spin-$\frac{1}{2}$) field; and $\mathsf{S}^{\mu\nu} = \mathsf{M}^{\mu\nu}$ for a vector (spin-1) field, such as the electromagnetic field.

Let us assume that the action is invariant under these transformations. Then taking into account the definition of the energy-momentum tensor, we find from Eq. (5.54)

$$f^\mu = -\frac{i}{2} \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)} \varepsilon\, \omega_{\nu\rho} (\mathsf{S}^{\nu\rho})_{rs} \phi_s - T^{\mu\nu}\, \varepsilon\, \omega_{\nu\rho} x^\rho, \tag{5.70}$$

and due to arbitrariness of the parameters $\omega_{\mu\nu}$ we obtain 6 independent conservation laws:

$$\partial_\mu M^{\mu\nu\rho} = 0, \quad \text{where} \quad M^{\mu\nu\rho} = -i \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)} (\mathsf{S}^{\nu\rho})_{rs} \phi_s - (T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu) \tag{5.71}$$

is the *(total) angular momentum tensor*. Its first term, which depends on intrinsic transformation properties of the field ϕ_r through the form of the generators $\mathsf{S}^{\mu\nu}$, is called the *spin tensor*, and the rest is the *orbital angular momentum tensor*.

5.4.3 Internal rotations and conserved currents

Consider internal transformations of a multi-component field $\Phi = (\phi_r)$ of the form

$$\Phi'(x) = \exp\left(i\varepsilon\lambda_a \mathsf{T}_a\right) \Phi(x) \approx \Phi(x) + i\varepsilon\lambda_a \mathsf{T}_a \Phi(x), \tag{5.72}$$

where the matrices T_a acting on the internal space are generators of a certain Lie group of internal symmetries. Keeping the spacetime coordinates fixed, $\delta x = 0$, we find, in components,

$$\delta\phi_r(x) = i\varepsilon\lambda_a (\mathsf{T}_a)_{rs} \phi_s(x) \quad \to \quad f^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)} \delta\phi_r = i\varepsilon\lambda_a \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)} (\mathsf{T}_a)_{rs} \phi_s. \tag{5.73}$$

If the transformation in question is a symmetry for arbitrary coefficients λ_a (assumed x-independent), we obtain independent Noether currents, one for each generator T_a:

$$\partial_\mu j^\mu_a = 0, \quad \text{where} \quad j^\mu_a = -i \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)} (\mathsf{T}_a)_{rs} \phi_s. \tag{5.74}$$

(The factor -1 has been included to match the usual conventions.) The corresponding Noether charges can be expressed with a help of the canonical momentum $\pi_r = \frac{\partial \mathscr{L}}{\partial(\partial_0 \phi_r)}$ as

$$Q_a = \int d^3x\, j^0_a = -i \int d^3x\, \pi_r (\mathsf{T}_a)_{rs} \phi_s. \tag{5.75}$$

A simple case to consider is that of a two-component real field, and internal two-dimensional rotations

$$\Phi'(x) = e^{i\varepsilon \mathsf{T}}\Phi(x), \quad \text{with} \quad \mathsf{T} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{5.76}$$

Instead of the components ϕ_1 and ϕ_2 it is advantageous to use a complex field φ and its complex conjugate φ^* defined via

$$\begin{pmatrix} \varphi \\ \varphi^* \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_1 - i\phi_2 \end{pmatrix} = \mathsf{U} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \text{where} \quad \mathsf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \quad \text{is unitary.} \tag{5.77}$$

This is a transformation of generalized coordinates within the field-theoretic Lagrangian formalism, which does not change the form of Euler-Lagrange equations, nor the form of Noether currents. In complex notation Eq. (5.72) reads

$$\begin{pmatrix} \varphi' \\ \varphi'^* \end{pmatrix} = \mathsf{U}\Phi' = \exp\left(i\varepsilon \mathsf{UTU}^\dagger\right)\mathsf{U}\Phi = \begin{pmatrix} e^{i\varepsilon}\varphi \\ e^{-i\varepsilon}\varphi^* \end{pmatrix} \approx \begin{pmatrix} \varphi + i\varepsilon\varphi \\ \varphi^* - i\varepsilon\varphi^* \end{pmatrix}, \quad \text{since} \quad \mathsf{UTU}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.78}$$

Hence we identify the variations $\delta\varphi = i\varepsilon\varphi$, $\delta\varphi^* = -i\varepsilon\varphi^*$, and the conserved current

$$j^\mu = -\frac{1}{\varepsilon} f^\mu = -i\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)}\varphi + i\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi^*)}\varphi^*. \tag{5.79}$$

(For a complex field, φ and φ^* are regarded as its independent 'components', and the Noether currents are obtained by summing over these components — just like in Eq. (5.54), where the sum runs over components enumerated by r.)

For example, the Lagrangian of a free complex Klein-Gordon field φ,

$$\mathscr{L} = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2\varphi^*\varphi, \tag{5.80}$$

is clearly invariant under phase rotations $\varphi'(x) = e^{i\lambda}\varphi(x)$, $\varphi'^*(x) = e^{-i\lambda}\varphi^*(x)$, and the corresponding current reads

$$j^\mu = -i\varphi\,\partial^\mu\varphi^* + i\varphi^*\partial^\mu\varphi. \tag{5.81}$$

This Noether current coincides (up to a constant prefactor) with the 'Klein-Gordon' current in Eq. (1.23).

Finally, let us remark that for internal transformations there is again a quick derivation of conserved currents that bypasses the general procedure of Section 5.4. Since spacetime transformations are not involved, the invariance of the action is tantamount to the invariance of the Lagrangian, whose variation reads

$$\mathscr{L}(\phi_r + \delta\phi_r, \partial_\mu\phi_r + \partial_\mu\delta\phi_r, x) - \mathscr{L}(\phi_r, \partial_\mu\phi_r, x) = \left(\frac{\partial \mathscr{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)}\right)\delta\phi_r + \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)}\delta\phi_r\right). \tag{5.82}$$

If this variation is zero, and the Euler-Lagrange equations hold, we obtain the continuity equation

$$\partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)}\delta\phi_r\right) = 0. \tag{5.83}$$

5.5 Exercises

Exercise 19. *Two coupled oscillators.* Consider an oscillating system with Lagrangian

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{M}{2}\dot{q}_1^2 + \frac{M}{2}\dot{q}_2^2 - \frac{\kappa_0}{2}q_1^2 - \frac{\kappa_0}{2}q_2^2 - \frac{\kappa_I}{2}(q_2 - q_1)^2, \qquad \kappa_0, \kappa_I > 0. \tag{5.84}$$

1. What is the system's Hamiltonian?
2. Find the normal modes and their frequencies.
3. Write general solution of the equations of motion.

Solution:

1. The Hamiltonian is obtained by determining canonical momenta, $p_n = \frac{\partial L}{\partial \dot{q}_n} = M\dot{q}_n$ ($n = 1, 2$), and, in this case, simply changing the sign of the potential term:

$$H(q_n, p_n) = \frac{p_1^2}{2M} + \frac{p_2^2}{2M} + \frac{1}{2}\vec{q}^T \mathsf{U}\,\vec{q}, \quad \text{where} \quad \mathsf{U} = \begin{pmatrix} \kappa_0 + \kappa_I & -\kappa_I \\ -\kappa_I & \kappa_0 + \kappa_I \end{pmatrix}, \quad \vec{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \tag{5.85}$$

2. We diagonalise the matrix U:

$$\mathsf{U}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \kappa_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \quad \mathsf{U}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (\kappa_0 + 2\kappa_I)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{5.86}$$

whence

$$\mathsf{U}\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{\mathsf{V}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\underbrace{\begin{pmatrix} \kappa_0 & 0 \\ 0 & \kappa_0 + 2\kappa_I \end{pmatrix}}_{\mathsf{K}} \quad \to \quad \mathsf{U} = \mathsf{V}\,\mathsf{K}\,\mathsf{V}^T \tag{5.87}$$

By comparison with Eq. (5.8) we identify

$$\mathsf{K} = M\,\Omega^2 \quad \to \quad \Omega = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\kappa_0/M} & 0 \\ 0 & \sqrt{(\kappa_0 + 2\kappa_I)/M} \end{pmatrix}. \tag{5.88}$$

3. General trajectory reads (see Eq. (5.14))

$$\vec{q}(t) = \sum_{k=1}^{2} \frac{\vec{v}_k}{\sqrt{2M\omega_k}}\Big(A_k e^{-i\omega_k t} + A_k^* e^{i\omega_k t}\Big), \quad \text{where} \quad \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{5.89}$$

and where $A_1, A_2 \in \mathbb{C}$ are arbitrary constants fixing the initial conditions.

Exercise 20. *Functional derivative of an integral functional.* Calculate a functional derivative of a (generalised) action functional

$$S[\phi_r] = \int d^D x\, \mathscr{L}\big(\phi_r(x), \partial_\mu\phi_r(x), \partial_\mu\partial_\nu\phi_r(x), x\big). \tag{5.90}$$

Solution:

$$\begin{aligned}
\frac{\delta S}{\delta \phi_s(y)} &= \int d^D x \left[\frac{\partial \mathscr{L}}{\partial \phi_r} \frac{\delta \phi_r(x)}{\delta \phi_s(y)} + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)} \frac{\delta(\partial_\mu \phi_r(x))}{\delta \phi_s(y)} + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \partial_\nu \phi_r)} \frac{\delta(\partial_\mu \partial_\nu \phi_r(x))}{\delta \phi_s(y)} \right] \\
&= \int d^D x \left[\frac{\partial \mathscr{L}}{\partial \phi_r} + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)} \partial_\mu + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \partial_\nu \phi_r)} \partial_\mu \partial_\nu \right] \delta(x-y) \delta_{rs} \\
&= \int d^D x \left[\frac{\partial \mathscr{L}}{\partial \phi_s} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_s)} + \partial_\mu \partial_\nu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \partial_\nu \phi_s)} \right] \delta(x-y) \\
&= \frac{\partial \mathscr{L}}{\partial \phi_s} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_s)} + \partial_\mu \partial_\nu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \partial_\nu \phi_s)} \bigg|_y ,
\end{aligned} \tag{5.91}$$

where y is substituted in each of the resulting terms in all functions in $\mathscr{L}$. It should be clear how to generalise this formula to the case of a function $\mathscr{L}$ depending on arbitrarily high derivatives of ϕ.

Exercise 21. *Klein-Gordon field in Lagrangian and Hamiltonian formalism.* Consider the Lagrangian density of a one-component real Klein-Gordon field

$$\mathscr{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2. \tag{5.92}$$

1. Derive the Euler-Lagrange equation of motion.
2. Find the canonical momentum field π and the Hamiltonian density $\mathscr{H}$.
3. Write Hamilton's canonical equations, and show that they combine to yield the Euler-Lagrange equation.

Solution:

1. By Eq. (5.35),

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \quad \to \quad \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathscr{L}}{\partial \phi} = \partial^\mu \partial_\mu \phi + m^2 \phi = 0 \tag{5.93}$$

(The equation of motion of the Klein-Gordon field is indeed the Klein-Gordon equation.)

2. By Eqs. (5.37) and (5.40),

$$\pi = \frac{\partial \mathscr{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi \quad \to \quad \mathscr{H} = \pi^2 - \mathscr{L} = \frac{1}{2}[\pi^2 + (\partial_i \phi)(\partial_i \phi) + m^2 \phi^2]. \tag{5.94}$$

3. By Eq. (5.44),

$$\begin{aligned}
\partial_t \phi &= \frac{\partial \mathscr{H}}{\partial \pi} = \pi, \\
\partial_t \pi &= -\frac{\partial \mathscr{H}}{\partial \phi} + \partial_i \frac{\partial \mathscr{H}}{\partial(\partial_i \phi)} = -m^2 \phi + \partial_i \partial_i \phi.
\end{aligned} \tag{5.95}$$

Taking time derivative of the first equation and substituting into the second we find

$$\partial_t^2 \phi = -m^2 \phi + \partial_i \partial_i \phi \quad \to \quad (\partial^\mu \partial_\mu + m^2)\phi = 0. \tag{5.96}$$

Exercise 22. *Dirac field Lagrangian and energy-momentum tensor.* Consider the Lagrangian density for the Dirac field

$$\mathscr{L}(\Psi,\bar{\Psi},\partial_\mu\Psi,\partial_\mu\bar{\Psi}) = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = i\bar{\psi}_\alpha(\gamma^\mu)_{\alpha\beta}\,\partial_\mu\psi_\beta - m\,\bar{\psi}_\alpha\psi_\alpha, \tag{5.97}$$

where the bispinor $\Psi = (\psi_\alpha)$ and its Dirac conjugate $\bar{\Psi} = (\bar{\psi}_\alpha)$, where $\bar{\psi}_\alpha = \psi^*_\beta\gamma^0_{\beta\alpha}$, are treated as independent four-component fields.

1. Derive the Euler-Lagrange equations of motion.
2. Show that orthochronous Lorentz transformations are symmetries of the action.
3. Determine the Dirac field's canonical energy-momentum tensor, and the total energy P_0.
4. Determine the conserved current and the integrated total charge corresponding to the 'phase' transformation $\Psi'(x) = e^{i\lambda}\Psi(x)$.

Solution:

1. Variations with respect to $\bar{\psi}_\alpha$ yield the Dirac equation,

$$\frac{\partial\mathscr{L}}{\partial\bar{\psi}_\alpha} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\bar{\psi}_\alpha)} = (i\gamma^\mu\partial_\mu\Psi)_\alpha - m\psi_\alpha - \partial_\mu(0) \quad\to\quad (i\gamma^\mu\partial_\mu - m)\Psi = 0, \tag{5.98}$$

while variations with respect to ψ_α yield its Dirac conjugate,

$$\frac{\partial\mathscr{L}}{\partial\psi_\alpha} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi_\alpha)} = -m\bar{\psi}_\alpha - \partial_\mu(i\bar{\Psi}\gamma^\mu)_\alpha \quad\to\quad i\partial_\mu\bar{\Psi}\gamma^\mu + m\bar{\Psi} = 0. \tag{5.99}$$

2. Taking into account the transformation properties of the Dirac spinors under Lorentz transformations $x'^\mu = L^\mu{}_\nu x^\nu$,

$$\Psi'(x') = S(L)\Psi(x) \quad,\quad \bar{\Psi}'(x') = \bar{\Psi}(x)S(L)^{-1}, \quad\text{where}\quad S(L)^{-1}\gamma^\mu S(L) = L^\mu{}_\nu\gamma^\nu, \tag{5.100}$$

we may verify that

$$\begin{aligned}\int d^4x'\,\bar{\Psi}'(x')\big(i\gamma^\mu\partial'_\mu - m\big)\Psi'(x') &= \int d^4x\,\bar{\Psi}(x)S(L)^{-1}\Big(i\gamma^\mu(\mathsf{L}^{-1})^\nu{}_\mu\partial_\nu - m\Big)S(L)\Psi(x)\\ &= \int d^4x\,\bar{\Psi}(x)\Big(iL^\mu{}_\rho\gamma^\rho(\mathsf{L}^{-1})^\nu{}_\mu\partial_\nu - m\Big)\Psi(x)\\ &= \int d^4x\,\bar{\Psi}(x)\big(i\gamma^\mu\partial_\mu - m\big)\Psi(x),\end{aligned} \tag{5.101}$$

where we have also used the relations

$$d^4x' = |\det\mathsf{L}|\,d^4x = d^4x \quad,\quad \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = (\mathsf{L}^{-1})^\nu{}_\mu\frac{\partial}{\partial x^\nu}. \tag{5.102}$$

3. By Eq. (5.60) the canonical energy-momentum tensor reads

$$T^\mu{}_\nu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi_\alpha)}\partial_\nu\psi_\alpha + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\bar{\psi}_\alpha)}\partial_\nu\bar{\psi}_\alpha - \delta^\mu_\nu\mathscr{L} = i\bar{\Psi}\gamma^\mu\partial_\nu\Psi - \delta^\mu_\nu\mathscr{L}. \tag{5.103}$$

(It is worth to note that $\mathscr{L} = 0$ on shell, i.e., for solutions of the equations of motion.)

The total energy is

$$P_0 = \int d^3x\,T^0{}_0 = \int d^3x\,\Big(i\bar{\Psi}\gamma^0\partial_0\Psi - \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi\Big) = \int d^3x\,\bar{\Psi}(-i\gamma^i\partial_i + m)\Psi. \tag{5.104}$$

4. The internal transformation

$$\Psi'(x) = e^{i\varepsilon}\Psi(x) \approx \Psi(x) + i\varepsilon\Psi(x)$$
$$\bar{\Psi}'(x) = e^{-i\varepsilon}\bar{\Psi}(x) \approx \bar{\Psi}(x) - i\varepsilon\bar{\Psi}(x) \tag{5.105}$$

is clearly a symmetry of the Lagrangian. From Eq. (5.54) we find the conserved current:

$$f^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi_\alpha)}\delta\psi_\alpha + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\bar{\psi}_\alpha)}\delta\bar{\psi}_\alpha = (i\bar{\Psi}\gamma^\mu)_\alpha\, i\varepsilon\psi_\alpha \quad \rightarrow \quad J^\mu = -\frac{1}{\varepsilon}f^\mu = \bar{\Psi}\gamma^\mu\Psi, \tag{5.106}$$

which coincides with the Dirac current of Eq. (3.32).

The corresponding total charge reads

$$Q = \int d^3x\, J^0 = \int d^3x\, \bar{\Psi}\gamma^0\Psi = \int d^3x\, \Psi^\dagger\Psi. \tag{5.107}$$

Chapter 6

Quantum many-body systems

Canonical quantisation promotes functions $(F, G, \ldots)$ of phase-space variables to operators $(\hat{F}, \hat{G}, \ldots)$, and Poisson brackets to commutators,

$$[\hat{F}, \hat{G}] = i\hbar\, \widehat{\{F, G\}}_{PB} \qquad \text{(in particular,} \quad [\hat{q}, \hat{p}] = i\hbar\, \widehat{\{q, p\}}_{PB} = i\hbar\, \hat{1}). \tag{6.1}$$

The classical Hamilton equations then become the quantum Heisenberg equations for time evolution of operators in Heisenberg picture:

$$\frac{d}{dt} F = \{F, H\}_{PB} \quad \to \quad \frac{d}{dt}\hat{F} = -\frac{i}{\hbar}[\hat{F}, \hat{H}]. \tag{6.2}$$

In quantum field theory, the classical fields $\phi(\boldsymbol{x}, t)$ (i.e., $\mathbb{C}$ or $\mathbb{R}$-valued functions, possibly with several components) are promoted to operator-valued functions $\hat{\phi}(\boldsymbol{x}, t)$. (The Heisenberg picture, where operators are time-dependent, is therefore rather natural.) To gain some intuition for these somewhat abstract objects let us first step back from continuous classical fields and discuss quantization of discrete systems of coupled oscillators, following the classical treatment in Chapter 5.1.

(In this chapter $\hbar$ and c will be shown explicitly.)

6.1 Quantum coupled oscillators

6.1.1 One-dimensional harmonic oscillator

The quantum version of the one-dimensional harmonic oscillator of Section 5.1.1 is described by the Hamiltonian operator

$$\hat{H} = \frac{\hat{p}^2}{2M} + \frac{1}{2} M\omega^2 \hat{q}^2. \tag{6.3}$$

The Hamiltonian can be factorised using creation and annihilation (or ladder) operators $\hat{a}^\dagger$ and $\hat{a}$ (quantum analogues of the classical complex variables z^* and z), defined by

$$\hat{a} = \frac{\hat{z}}{\sqrt{\hbar}} = \sqrt{\frac{M\omega}{2\hbar}}\hat{q} + \frac{i\hat{p}}{\sqrt{2M\hbar\omega}}, \quad \text{and satisfying} \quad [\hat{a}, \hat{a}^\dagger] = \frac{1}{\hbar}[\hat{z}, \widehat{z^*}] = \frac{i\hbar}{\hbar} \underbrace{\widehat{\{z, z^*\}}_{PB}}_{-i} = 1. \tag{6.4}$$

One finds

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger \hat{a} + \hat{a}\,\hat{a}^\dagger\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{6.5}$$

where the term $\frac{1}{2}\hbar\omega$ (the ground state energy), which is not present in the classical Hamiltonian $H(z,z^*) = \omega z^* z$, appears due to non-commutativity of the operators $\hat{q}$ and $\hat{p}$. However, it does not affect the Heisenberg equations of motion, which read

$$\frac{d}{dt}\hat{a} = -\frac{i}{\hbar}[\hat{a},\hat{H}] = -i\omega[\hat{a},\hat{a}^\dagger]\hat{a} = -i\omega\hat{a} \quad \rightarrow \quad \hat{a}(t) = \hat{a}(0)e^{-i\omega t} \quad , \quad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t}. \tag{6.6}$$

The Heisenberg-picture position operator $\hat{q}(t)$ is then retrieved from $\hat{a}(t)$ and $\hat{a}^\dagger(t)$:

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2M\omega}}\Big(\hat{a}(t) + \hat{a}^\dagger(t)\Big) = \sqrt{\frac{\hbar}{2M\omega}}\Big(\hat{a}(0)e^{-i\omega t} + \hat{a}^\dagger(0)e^{i\omega t}\Big). \tag{6.7}$$

Similarly for the momentum operator $\hat{p}(t) = \frac{\sqrt{2M\hbar\omega}}{2i}\big(\hat{a}(t) - \hat{a}^\dagger(t)\big)$.

The eigenstates of $\hat{H}$ are enumerated by $n = 0, 1, 2, \ldots$:

$$\hat{H}\,|n\rangle = E_n\,|n\rangle\,, \quad \text{where} \quad E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad \text{and} \quad |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle\,. \tag{6.8}$$

The ground state $|0\rangle$ is annihilated by the annihilation operator $\hat{a}$,

$$\hat{a}\,|0\rangle = 0 \quad \rightarrow \quad \langle q|\left(\sqrt{\frac{M\omega}{2\hbar}}\hat{q} + \frac{i\hat{p}}{\sqrt{2M\hbar\omega}}\right)|0\rangle = \left(\sqrt{\frac{M\omega}{2\hbar}}q + \sqrt{\frac{\hbar}{2M\omega}}\frac{\partial}{\partial q}\right)\psi_0(q) = 0, \tag{6.9}$$

the latter equation yielding, by a straightforward integration, the ground state wave-function

$$\psi_0(q) = \left(\frac{M\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{M\omega}{2\hbar}q^2\right). \tag{6.10}$$

6.1.2 Many oscillators

Let us consider a quantum version of the classical system of coupled harmonic oscillators treated in Section 5.1.2. The Hamiltonian operator

$$\hat{H} = \sum_{n=1}^{N}\frac{\hat{p}_n^2}{2M} + \frac{1}{2}\sum_{n,n'=1}^{N} U_{nn'}\hat{q}_n\hat{q}_{n'} \tag{6.11}$$

can again be 'decoupled' by introducing normal coordinates, i.e., operators $\hat{\eta}_k$ (and the conjugate momenta $\hat{\rho}_k$) such that

$$\hat{\vec{q}} = \mathsf{V}\,\hat{\vec{\eta}}, \qquad \text{where} \qquad \mathsf{V}^T\mathsf{V} = \mathbb{I} \quad , \quad \mathsf{U} = \mathsf{V}\,M\Omega^2\,\mathsf{V}^T \quad , \quad \Omega = \mathrm{diag}(\omega_1, \ldots, \omega_N). \tag{6.12}$$

To each mode $k = 1, \ldots, N$ corresponds a creation operator $\hat{a}_k^\dagger$ and an annihilation operator $\hat{a}_k$ (constructed as in Eq. (6.4) with $\hat{q} \rightarrow \hat{\eta}_k$, $\hat{p} \rightarrow \hat{\rho}_k$, $\omega \rightarrow \omega_k$), which satisfy the commutation relations

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'} \quad , \quad [\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0, \tag{6.13}$$

and in terms of which the Hamiltonian reads

$$\hat{H} = \sum_k \hat{H}_k = \sum_k\left(\frac{\hat{\rho}_k^2}{2M} + \frac{M}{2}\omega_k^2\hat{\eta}_k^2\right) = \sum_k \hbar\omega_k\left(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}\right). \tag{6.14}$$

(Compare with the classical expressions (5.12) and (5.13).)

Since $\hat{H}$ is a sum of independent Hamiltonians $\hat{H}_k$ corresponding to the individual normal modes, it is easy to obtain the full set of eigenstates of the system, enumerated by N numbers $n_k = 0, 1, 2, \ldots$:

$$\hat{H}\,|\ldots, n_k, \ldots\rangle = (E^{(1)}_{n_1} + \cdots + E^{(N)}_{n_N})\,|\ldots, n_k, \ldots\rangle\,, \tag{6.15}$$

where the energy levels of the kth mode are

$$E^{(k)}_{n_k} = \hbar\omega_k\left(n_k + \frac{1}{2}\right), \qquad \text{and} \qquad |\ldots, n_k, \ldots\rangle = \prod_{k=1}^{N}\frac{(\hat{a}^\dagger_k)^{n_k}}{\sqrt{n_k!}}\,|0\rangle \tag{6.16}$$

are the eigenstates of the system, in which the individual modes are excited to levels $n_1, n_2, \ldots, n_N$. $|0\rangle \equiv |0, \ldots, 0\rangle$ is the *vacuum state* — the lowest energy state of the system with energy $E_0 = \frac{\hbar}{2}\sum_k \omega_k$.

The operators of normal coordinates $\hat{\eta}_k$ evolve in time according to Eq. (6.7) (with the replacements $\hat{q} \to \hat{\eta}_k$, $\omega \to \omega_k$). A general solution of the Heisenberg equations of motion for the position (or rather displacement) operators $\hat{q}_n$ is then, analogously to the classical case, Eq. (5.14),

$$\hat{\vec{q}}(t) = \sum_k \vec{v}_k\hat{\eta}_k(t) = \sum_k \frac{\vec{v}_k}{\sqrt{2M\omega_k/\hbar}}\Big(\hat{a}_k(0)e^{-i\omega_k t} + \hat{a}^\dagger_k(0)e^{i\omega_k t}\Big), \tag{6.17}$$

where $\vec{v}_k$ are columns of the matrix V (the 'shapes' of the normal modes).

6.1.3 Vacuum state

The vacuum $|0\rangle$ is the state that is annihilated by all annihilation operators, $\hat{a}_k\,|0\rangle = 0$, and so it can be written explicitly as a product over all modes k of the single-oscillator ground state wave-functions (6.10) (with $q \to \eta_k$, $\omega \to \omega_k$):

$$\psi_0(\vec{\eta}) = \mathcal{N}\exp\left(-\frac{M}{2\hbar}\sum_k \omega_k\eta_k^2\right) = \mathcal{N}\exp\left(-\frac{M}{2\hbar}\vec{\eta}^T\Omega\,\vec{\eta}\right), \quad \text{where} \quad \mathcal{N} \equiv \left(\prod_k \frac{M\omega_k}{\pi\hbar}\right)^{1/4}. \tag{6.18}$$

Expressed in the original coordinates q_n, via $\vec{\eta} = \mathsf{V}^T\vec{q}$, the vacuum wave-function has the form of a multivariate Gaussian state

$$\psi_0(\vec{q}) = \mathcal{N}\exp\left(-\frac{M}{2\hbar}\vec{q}^T\mathsf{V}\,\Omega\,\mathsf{V}^T\vec{q}\right) = \mathcal{N}\exp\Bigg(-\frac{1}{2\hbar}\sum_{n,n'}A_{nn'}q_n q_{n'}\Bigg), \tag{6.19}$$

where the matrix

$$\mathsf{A} = (A_{nn'}) = M\,\mathsf{V}\,\Omega\,\mathsf{V}^T = (M\,\mathsf{U})^{1/2} \quad \text{is symmetric positive-definite.} \tag{6.20}$$

The structure of the vacuum state, presented by the matrix A, is related to the structure of the oscillatory network, i.e., the potential energy matrix U, in a straightforward way via the square root in Eq. (6.20). Hence, if U possesses a certain symmetry (i.e., there exists a matrix P such that $\mathsf{P}\mathsf{U}\mathsf{P}^{-1} = \mathsf{U}$, or $[\mathsf{P}, \mathsf{U}] = 0$), the "ground state matrix" A exhibits this symmetry as well: $[\mathsf{P}, \mathsf{A}] = 0$. For example, the linear chain of Section 5.1.3 is invariant under (discrete) translations with $P_{nn'} = \delta_{n+1,n'}$ that act by shifting the index n: $(\mathsf{P}\vec{q})_n = \sum_{n'}\delta_{n+1,n'}q_{n'} = q_{n+1}$.

An important observation is that although in normal coordinates the vacuum state is a product state (it is a product of the individual modes' ground states), it is a correlated (entangled) state in the original coordinates q_n (see Exercise 24 for an example). These *vacuum correlations* are captured by the matrix A:

$$\langle 0| \hat{q}_n \hat{q}_{n'} |0\rangle = \frac{\hbar}{2}(\mathsf{A}^{-1})_{nn'} \quad , \quad \langle 0| \hat{q}_n |0\rangle = 0, \tag{6.21}$$

as shown in Exercise 23 (the second formula is a trivial consequence of the vacuum wave-function being symmetric under $\vec{q} \to -\vec{q}$).

6.2 Field theory for non-relativistic many-particle systems

In condensed matter physics one typically studies non-relativistic quantum systems consisting of many indistinguishable particles, e.g., a gas of electrons in a solid state sample. If interparticle interactions are neglected, the N-particle wave-function Ψ follows the Schrödinger equation of the form

$$i\hbar \frac{\partial \Psi}{\partial t} = (H_{\boldsymbol{x}_1} + \cdots + H_{\boldsymbol{x}_N})\Psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, t), \quad \text{where} \quad H_{\boldsymbol{x}} = -\frac{\hbar^2}{2m}\Delta_{\boldsymbol{x}} + V(\boldsymbol{x}) \tag{6.22}$$

is a single-particle Hamiltonian. The multi-particle wave-function $\Psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ is either totally symmetric (in the case of boson), or totally antisymmetric (in the case of fermions). The 'external' (or 'background') potential V is identical for each particle (e.g., the electric potential of an atomic lattice).

In this section we will provide a quantum-field-theoretical description of this many-body system. It will serve as an intermediate step between non-relativistic quantum mechanics and relativistic quantum field theory, which will be of our main interest from Chapter 7 onwards.

We start by observing that the single-particle Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H_{\boldsymbol{x}} \psi(\boldsymbol{x}, t) \tag{6.23}$$

can be viewed as an equation of motion of a classical field theory with Lagrangian density (ψ and ψ^* treated as independent fields)

$$\mathscr{L} = i\hbar\psi^* \partial_t \psi - \frac{\hbar^2}{2m}(\partial_i \psi^*)(\partial_i \psi) - V(\boldsymbol{x})\psi^* \psi \tag{6.24}$$

(see Exercise 25). The canonical momenta and the field Hamiltonian read

$$\pi = \frac{\partial \mathscr{L}}{\partial(\partial_t \psi)} = i\hbar\psi^* \quad , \quad \pi^* = \frac{\partial \mathscr{L}}{\partial(\partial_t \psi^*)} = 0, \tag{6.25}$$

and

$$H = \int d^3x \, (\pi \partial_t \psi - \mathscr{L}) = \int d^3x \left(\frac{\hbar^2}{2m}(\partial_i \psi^*)(\partial_i \psi) + V(\boldsymbol{x})\psi^* \psi \right) = \int d^3x \, \psi^*(\boldsymbol{x}) \, H_{\boldsymbol{x}} \, \psi(\boldsymbol{x}). \tag{6.26}$$

Quantising the 'Schrödinger field' $\psi(\boldsymbol{x})$, we shall obtain a powerful method to handle the corresponding multi-particle system. This process of promoting a single-particle quantum-mechanical wave-function (viewed as a classical field) into an $\boldsymbol{x}$-dependent quantum operator $\hat{\psi}(\boldsymbol{x})$ is called the *second quantisation* (the *first quantisation* being the transition from the single-particle classical mechanics to quantum mechanics with Schrödinger equation (6.23)).

In passing let us note that interparticle interactions, for example, between electrons of the electron gas, often play an important role, and a large part of condensed matter physics is devoted to understanding their effects. We will discuss interacting systems at least briefly in Section 6.2.3 to see that the field-theoretic formalism extends nicely to this more general scenario.

6.2.1 Bosonic systems

Since $\psi^* = \frac{1}{i\hbar}\pi$ and $\pi^* = 0$, the field theory with Hamiltonian (6.26) features effectively only two canonical fields, ψ and π (or equivalently ψ and ψ^*). For *bosonic* systems we promote their canonical Poisson brackets

$$\{\psi(\boldsymbol{x}), \pi(\boldsymbol{y})\}_{PB} = \delta(\boldsymbol{x}-\boldsymbol{y}) \quad \text{and} \quad \{\psi(\boldsymbol{x}), \psi(\boldsymbol{y})\}_{PB} = \{\pi(\boldsymbol{x}), \pi(\boldsymbol{y})\}_{PB} = 0 \tag{6.27}$$

to equal-time *commutation* relations

$$[\hat{\psi}(\boldsymbol{x},t), \hat{\pi}(\boldsymbol{y},t)] = i\hbar\,\delta(\boldsymbol{x}-\boldsymbol{y}), \quad \text{or} \quad [\hat{\psi}(\boldsymbol{x},t), \hat{\psi}^\dagger(\boldsymbol{y},t)] = \delta(\boldsymbol{x}-\boldsymbol{y}), \tag{6.28}$$

and

$$[\hat{\psi}(\boldsymbol{x},t), \hat{\psi}(\boldsymbol{y},t)] = [\hat{\psi}^\dagger(\boldsymbol{x},t), \hat{\psi}^\dagger(\boldsymbol{y},t)] = 0 \tag{6.29}$$

between operator-valued fields. We are working in the Heisenberg picture, in which the field operator depends on time as

$$\hat{\psi}(\boldsymbol{x},t) = e^{\frac{i}{\hbar}t\hat{H}}\hat{\psi}(\boldsymbol{x},0)e^{-\frac{i}{\hbar}t\hat{H}}, \quad \hat{\psi}(\boldsymbol{x}) \equiv \hat{\psi}(\boldsymbol{x},0), \quad \text{where} \quad \hat{H} = \int d^3x\,\hat{\psi}^\dagger(\boldsymbol{x})\,H_{\boldsymbol{x}}\,\hat{\psi}(\boldsymbol{x}) \tag{6.30}$$

is an operator analogue of the total Hamiltonian (6.26) of the classical field theory. $\hat{H}$ is time-independent, $\frac{d}{dt}\hat{H} = -\frac{i}{\hbar}[\hat{H},\hat{H}] = 0$, since we assume that the potential $V(\boldsymbol{x})$ (and hence $H_{\boldsymbol{x}}$) does not depend explicitly on time.

Heisenberg's equation of motion for the quantum field reads

$$\begin{aligned} i\hbar\frac{\partial\hat{\psi}}{\partial t}(\boldsymbol{x},t) &= [\hat{\psi}(\boldsymbol{x},t), \hat{H}(t)] \\ &= \int d^3y\,[\hat{\psi}(\boldsymbol{x},t), \hat{\psi}^\dagger(\boldsymbol{y},t)H_{\boldsymbol{y}}\hat{\psi}(\boldsymbol{y},t)] \\ &= \int d^3y\,\delta(\boldsymbol{x}-\boldsymbol{y})\,H_{\boldsymbol{y}}\hat{\psi}(\boldsymbol{y},t) \\ &= H_{\boldsymbol{x}}\hat{\psi}(\boldsymbol{x},t). \end{aligned} \tag{6.31}$$

This equation is nothing but the Schrödinger equation for an unknown function $\hat{\psi}(\boldsymbol{x},t)$, which is now operator-valued.

To solve it we employ the eigenstates of the (first quantised, or single-particle) Hamiltonian $H_{\boldsymbol{x}}$, i.e., the complete set of solutions of the one-particle stationary Schrödinger equation:

$$H_{\boldsymbol{x}}u_k(\boldsymbol{x}) = \epsilon_k u_k(\boldsymbol{x}) \quad , \quad \int d^3x\,u_k^*(\boldsymbol{x})u_{k'}(\boldsymbol{x}) = \delta_{kk'} \quad , \quad \sum_k u_k^*(\boldsymbol{x})u_k(\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y}), \tag{6.32}$$

where $k = 1, 2, \ldots$, and ϵ_k are (possibly degenerate) one-particle energy levels. In this fixed basis we expand the quantum field,

$$\hat{\psi}(\boldsymbol{x},t) = \int d^3y\,\hat{\psi}(\boldsymbol{y},t)\,\delta(\boldsymbol{y}-\boldsymbol{x}) = \sum_k \underbrace{\int d^3y\,\hat{\psi}(\boldsymbol{y},t)u_k^*(\boldsymbol{y})}_{\hat{a}_k(t)}\,u_k(\boldsymbol{x}) = \sum_k \hat{a}_k(t)u_k(\boldsymbol{x}), \tag{6.33}$$

so the operator nature is now carried by the time-dependent expansion coefficients $\hat{a}_k(t)$, $\hat{a}_k^\dagger(t)$ ($u_k(\boldsymbol{x})$ are ordinary complex-valued functions). Plugging this expansion into Eq. (6.31) we have

$$\sum_k i\hbar\frac{d\hat{a}_k}{dt}u_k(\boldsymbol{x}) = \sum_k \hat{a}_k(t)\,H_{\boldsymbol{x}}u_k(\boldsymbol{x}) = \sum_k \epsilon_k\hat{a}_k(t)u_k(\boldsymbol{x}), \tag{6.34}$$

and due to the orthogonality (and hence linear independence) of the eigenstates $u_k(\boldsymbol{x})$ we can omit the sum, and obtain for each k separately

$$i\hbar\frac{d\hat{a}_k}{dt} = \epsilon_k \hat{a}_k(t) \qquad \to \qquad \hat{a}_k(t) = e^{-\frac{i}{\hbar}t\,\epsilon_k}\hat{a}_k, \quad \text{where} \quad \hat{a}_k \equiv \hat{a}_k(0). \tag{6.35}$$

The operators $\hat{a}_k^\dagger$ will turn out to be the creation operators corresponding to the 'modes' (or orbitals) u_k, and $\hat{a}_k$ the respective annihilation operators. To see this let us first use the commutation relations (6.28) to calculate the equal-time commutators

$$\begin{aligned}
[\hat{a}_k(t), \hat{a}_{k'}^\dagger(t)] &= \int d^3x\, d^3y\, [\hat{\psi}(\boldsymbol{x}, t), \hat{\psi}^\dagger(\boldsymbol{y}, t)]\, u_k^*(\boldsymbol{x}) u_{k'}(\boldsymbol{y}) \\
&= \int d^3x\, d^3y\, \delta(\boldsymbol{x} - \boldsymbol{y}) u_k^*(\boldsymbol{x}) u_{k'}(\boldsymbol{y}) \\
&= \int d^3x\, u_k^*(\boldsymbol{x}) u_{k'}(\boldsymbol{x}) \\
&= \delta_{kk'}.
\end{aligned} \tag{6.36}$$

Similarly, using (6.29), we find

$$[\hat{a}_k(t), \hat{a}_{k'}(t)] = [\hat{a}_k^\dagger(t), \hat{a}_{k'}^\dagger(t)] = 0. \tag{6.37}$$

Now let us define the vacuum state $|0\rangle$ implicitly by the requirements

$$\langle 0|0\rangle = 1 \quad \text{and} \quad \hat{a}_k\,|0\rangle = 0 \quad (\forall k), \tag{6.38}$$

and calculate

$$\langle 0|\,\hat{\psi}(\boldsymbol{x})\,\hat{a}_k^\dagger\,|0\rangle = \sum_{k'} u_{k'}(\boldsymbol{x})\,\langle 0|\,[\hat{a}_{k'}, \hat{a}_k^\dagger]\,|0\rangle = \sum_{k'} u_{k'}(\boldsymbol{x})\,\langle 0|\,\delta_{k'k}\,|0\rangle = u_k(\boldsymbol{x}). \tag{6.39}$$

$\hat{a}_k^\dagger\,|0\rangle$ can be interpreted as a one-particle state whose $\boldsymbol{x}$-representation, produced by the action of $\langle 0|\,\hat{\psi}(\boldsymbol{x}) \equiv \langle \boldsymbol{x}|$, coincides with the eigenfunction $u_k(\boldsymbol{x})$. (Similarly, since $\langle 0|\,\hat{\psi}(\boldsymbol{x})\,\hat{\psi}^\dagger(\boldsymbol{y})\,|0\rangle = \delta(\boldsymbol{x} - \boldsymbol{y})$ is the wave-function of a particle localized at position $\boldsymbol{y}$, the operator $\hat{\psi}^\dagger(\boldsymbol{y})$ acting on the vacuum "creates" a particle at position $\boldsymbol{y}$.) Further applications of the creation operators give rise to (normalized) states of the form

$$|n_1, n_2, \ldots\rangle = \frac{(\hat{a}_1^\dagger)^{n_1}}{\sqrt{n_1!}}\frac{(\hat{a}_2^\dagger)^{n_2}}{\sqrt{n_2!}}\ldots|0\rangle\,, \tag{6.40}$$

which span the so-called bosonic Fock space — the Hilbert space of a system (a grand-canonical ensemble) with arbitrary number of indistinguishable bosons. These states are eigenstates of the *particle number operators*

$$\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k, \qquad \text{which satisfy} \qquad [\hat{n}_k, \hat{n}_{k'}] = 0 \quad \text{and} \quad [\hat{n}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}\hat{a}_k^\dagger \quad (\text{no sum over } k). \tag{6.41}$$

Indeed, since $\hat{n}_k\,|0\rangle = 0$, we easily calculate

$$\hat{n}_k\,(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}\ldots|0\rangle = (\hat{a}_1^\dagger)^{n_1}\ldots[\hat{n}_k, (\hat{a}_k^\dagger)^{n_k}]\ldots|0\rangle = n_k\,(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}\ldots|0\rangle\,. \tag{6.42}$$

Note that the particle number operators are time-independent:

$$\hat{n}_k(t) = \hat{a}_k^\dagger(t)\,\hat{a}_k(t) = e^{\frac{i}{\hbar}t\,\epsilon_k}\hat{a}_k^\dagger\,e^{-\frac{i}{\hbar}t\,\epsilon_k}\hat{a}_k = \hat{n}_k. \tag{6.43}$$

An N-particle state that contains n_1 particles in mode u_1, n_2 particles in mode u_2, etc. $(n_1 + n_2 + \ldots = N)$ can be written as

$$|n_1, n_2, \ldots\rangle = C\, \hat{a}^\dagger_{k_1} \ldots \hat{a}^\dagger_{k_N} |0\rangle\,, \quad \text{where} \quad C \equiv \frac{1}{\sqrt{n_1! n_2! \ldots}}, \tag{6.44}$$

and $k_1 = \ldots = k_{n_1} = 1$, $k_{n_1+1} = \ldots = k_{n_1+n_2} = 2$, etc. The corresponding N-particle wave-function reads, by Exercise 26,

$$\frac{1}{\sqrt{N!}} \langle 0| \hat{\psi}(\boldsymbol{x}_1) \ldots \hat{\psi}(\boldsymbol{x}_N) |n_1, n_2, \ldots\rangle = \frac{C}{\sqrt{N!}} \sum_{\sigma \in \mathcal{S}_N} u_{k_1}(\boldsymbol{x}_{\sigma(1)}) \ldots u_{k_N}(\boldsymbol{x}_{\sigma(N)}), \tag{6.45}$$

where $\mathcal{S}_N$ is the group of permutations of the set $\{1, \ldots, N\}$. This wave-function is totally symmetric since it is constructed with the commuting (bosonic) operators $\hat{\psi}(\boldsymbol{x})$.

The field Hamiltonian operator expressed in terms of $\hat{a}_k$ and $\hat{a}^\dagger_k$ reads

$$\begin{aligned}
\hat{H}(t) &= \int d^3x\, \hat{\psi}^\dagger(\boldsymbol{x}, t) H_{\boldsymbol{x}} \hat{\psi}(\boldsymbol{x}, t) \\
&= \int d^3x\, \hat{\psi}^\dagger(\boldsymbol{x}, t) H_{\boldsymbol{x}} \sum_k \hat{a}_k(t) u_k(\boldsymbol{x}) \\
&= \sum_k \epsilon_k \int d^3x\, \hat{\psi}^\dagger(\boldsymbol{x}, t) u_k(\boldsymbol{x}) \hat{a}_k(t) \\
&= \sum_k \epsilon_k\, \hat{a}^\dagger_k(t) \hat{a}_k(t) \\
&= \sum_k \epsilon_k \hat{n}_k
\end{aligned} \tag{6.46}$$

(which is indeed a time-independent operator). The total energy of any multi-particle state $|n_1, n_2, \ldots\rangle$ is therefore easy to find:

$$\hat{H}\, |n_1, n_2, \ldots\rangle = \Big(\sum_k \epsilon_k n_k\Big) |n_1, n_2, \ldots\rangle \tag{6.47}$$

Finally, let us show that the time evolution of the N-particle wave-function of Eq. (6.22) is implied by the time evolution of the quantum field $\hat{\psi}(\boldsymbol{x}, t)$ satisfying Eq. (6.31). To this end take an arbitrary fixed state $|\Psi\rangle$ in the Fock space, and identify

$$\Psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, t) = \frac{1}{\sqrt{N!}} \langle 0| \hat{\psi}(\boldsymbol{x}_1, t) \ldots \hat{\psi}(\boldsymbol{x}_N, t) |\Psi\rangle\,. \tag{6.48}$$

The wave-function thus constructed then satisfies

$$\begin{aligned}
i\hbar \frac{\partial}{\partial t} \Psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, t) &= \frac{i\hbar}{\sqrt{N!}} \langle 0| \sum_{p=1}^N \hat{\psi}(\boldsymbol{x}_1, t) \ldots \frac{\partial \hat{\psi}(\boldsymbol{x}_p, t)}{\partial t} \ldots \hat{\psi}(\boldsymbol{x}_N, t) |\Psi\rangle \\
&= \frac{1}{\sqrt{N!}} \langle 0| \sum_{p=1}^N \hat{\psi}(\boldsymbol{x}_1, t) \ldots H_{\boldsymbol{x}_p} \hat{\psi}(\boldsymbol{x}_p, t) \ldots \hat{\psi}(\boldsymbol{x}_N, t) |\Psi\rangle \\
&= \Big(\sum_{p=1}^N H_{\boldsymbol{x}_p}\Big) \Psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, t),
\end{aligned} \tag{6.49}$$

which is nothing but the N-particle Schrödinger equation (6.22).

6.2.2 Fermionic systems

For fermions the N-particle wave-function

$$\Psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, t) = \frac{1}{\sqrt{N!}} \langle 0| \hat{\psi}(\boldsymbol{x}_1, t) \ldots \hat{\psi}(\boldsymbol{x}_N, t) |\Psi\rangle \tag{6.50}$$

must be totally antisymmetric to comply with the Pauli exclusion principle. This can be achieved by postulating equal-time *anticommutation* relations

$$\{\hat{\psi}(\boldsymbol{x}, t), \hat{\psi}^\dagger(\boldsymbol{y}, t)\} = \delta(\boldsymbol{x} - \boldsymbol{y}) \quad , \quad \{\hat{\psi}(\boldsymbol{x}, t), \hat{\psi}(\boldsymbol{y}, t)\} = \{\hat{\psi}^\dagger(\boldsymbol{x}, t), \hat{\psi}^\dagger(\boldsymbol{y}, t)\} = 0. \tag{6.51}$$

This approach to quantization of classical Poisson brackets is different from our experience with quantum mechanics, where the correspondence principle always postulates commutation relations between quantum operators. Yet, in quantum field theory, quantizing with anticommutators is appropriate for all fermionic systems.

Using the anticommutator Leibniz role $[A, BC] = \{A, B\}C - B\{A, C\}$, the Heisenberg equation again turns into the Schrödinger equation for $\hat{\psi}(\boldsymbol{x}, t)$:

$$\begin{aligned} i\hbar \frac{\partial \hat{\psi}}{\partial t}(\boldsymbol{x}, t) &= \int d^3y \left[\hat{\psi}(\boldsymbol{x}, t), \hat{\psi}^\dagger(\boldsymbol{y}, t) H_{\boldsymbol{y}} \hat{\psi}(\boldsymbol{y}, t)\right] \\ &= \int d^3y \, \{\hat{\psi}(\boldsymbol{x}, t), \hat{\psi}^\dagger(\boldsymbol{y}, t)\} \, H_{\boldsymbol{y}} \hat{\psi}(\boldsymbol{y}, t) \\ &= H_{\boldsymbol{x}} \hat{\psi}(\boldsymbol{x}, t). \end{aligned} \tag{6.52}$$

Its solutions can again be found in the form

$$\hat{\psi}(\boldsymbol{x}, t) = \sum_k \hat{b}_k(t) u_k(\boldsymbol{x}) \quad , \quad \hat{b}_k(t) = e^{-\frac{i}{\hbar} t \, \epsilon_k} \hat{b}_k, \tag{6.53}$$

where u_k are the eigenstates of $H_{\boldsymbol{x}}$ from Eq. (6.32).

Similarly as in the bosonic case one can show that the fermionic creation and annihilation operators $\hat{b}_k^\dagger$ and $\hat{b}_k$ satisfy the anticommutation relations

$$\{\hat{b}_k(t), \hat{b}_{k'}^\dagger(t)\} = \delta_{kk'}, \quad \{\hat{b}_k(t), \hat{b}_{k'}(t)\} = \{\hat{b}_k^\dagger(t), \hat{b}_{k'}^\dagger(t)\} = 0. \tag{6.54}$$

Since $(\hat{b}_k^\dagger)^2 = \frac{1}{2}\{\hat{b}_k^\dagger, \hat{b}_k^\dagger\} = 0$, the (normalised) states

$$|n_1, n_2, \ldots\rangle = (\hat{b}_1^\dagger)^{n_1} (\hat{b}_2^\dagger)^{n_2} \ldots |0\rangle \quad \text{have} \quad n_k \in \{0, 1\}. \tag{6.55}$$

They span the fermionic Fock space. Alternatively, for $N = \sum_k n_k$, we can write

$$|n_1, n_2, \ldots\rangle = \hat{b}_{k_N}^\dagger \ldots \hat{b}_{k_1}^\dagger \, |0\rangle \,, \tag{6.56}$$

where $k_1, \ldots, k_N$ is a list of occupied modes. (Note that in the fermionic case, reordering of the creation operators can result in a sign change.) The corresponding N-particle fermionic wave-function reads (by Exercise 26)

$$\frac{1}{\sqrt{N!}} \langle 0| \hat{\psi}(\boldsymbol{x}_1) \ldots \hat{\psi}(\boldsymbol{x}_N) |n_1, n_2, \ldots\rangle = \frac{1}{\sqrt{N!}} \sum_{\sigma \in \mathcal{S}_N} (\operatorname{sgn} \sigma) \, u_{k_1}(\boldsymbol{x}_{\sigma(1)}) \ldots u_{k_N}(\boldsymbol{x}_{\sigma(N)}). \tag{6.57}$$

It is antisymmetric due to the anticommuting nature of the (fermionic) operators $\hat{\psi}(\boldsymbol{x})$.

The N-particle Schrödinger equation (6.22) follows from the equation (6.52) for quantum field $\hat{\psi}(\boldsymbol{x}, t)$ in the same way as in the bosonic case, Eq. (6.49).

6.2.3 Interparticle interactions

An N-body quantum-mechanical Hamiltonian that takes into account a two-body interparticle interaction potential V_I reads

$$\sum_{p=1}^{N} H_{\boldsymbol{x}_p} + \frac{1}{2}\sum_{p\neq p'} V_I(\boldsymbol{x}_p,\boldsymbol{x}_{p'}), \quad \text{where} \quad H_{\boldsymbol{x}} = -\frac{\hbar^2}{2m}\Delta_{\boldsymbol{x}} + V_0(\boldsymbol{x}) \quad \text{and} \quad V_I(\boldsymbol{x},\boldsymbol{y}) = V_I(\boldsymbol{y},\boldsymbol{x}). \tag{6.58}$$

For example, we could have $V_I(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{|\boldsymbol{x}-\boldsymbol{y}|}$ describing the Coulomb interaction between electrons. Standard quantum mechanics provides the N-particle Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(\sum_{p=1}^{N} H_{\boldsymbol{x}_p} + \frac{1}{2}\sum_{p\neq p'} V_I(\boldsymbol{x}_p,\boldsymbol{x}_{p'})\right)\Psi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N,t) \tag{6.59}$$

as a description of system's dynamics.

Alternatively, this interacting many-body system can be described by a field theory (see Exercise 25) with classical Lagrangian $L = L_0 + L_I$, where

$$L_0 = \int d^3x\,\mathscr{L}_0 \qquad \text{and} \qquad L_I = -\frac{1}{2}\int d^3x\,d^3y\,\psi^*(\boldsymbol{x},t)\psi^*(\boldsymbol{y},t)V_I(\boldsymbol{x},\boldsymbol{y})\psi(\boldsymbol{y},t)\psi(\boldsymbol{x},t) \tag{6.60}$$

are, respectively, the (local) 'free' Lagrangian with density $\mathscr{L}_0$ given by Eq. (6.24), and the (non-local) 'interacting' Lagrangian. The corresponding total Hamiltonian

$$H = \int d^3x\,\pi\partial_t\psi - L_0 - L_I = \int d^3x\,\psi^*(\boldsymbol{x})\,H_{\boldsymbol{x}}\,\psi(\boldsymbol{x}) - L_I, \tag{6.61}$$

after quantization of the field $\psi(\boldsymbol{x},t)$ using commutation relations (6.28) for bosons, or anticommutation relations (6.51) for fermions, becomes the operator

$$\hat{H} = \int d^3x\,\hat{\psi}^\dagger(\boldsymbol{x})\,H_{\boldsymbol{x}}\,\hat{\psi}(\boldsymbol{x}) + \frac{1}{2}\int d^3x\,d^3y\,\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{y})V_I(\boldsymbol{x},\boldsymbol{y})\hat{\psi}(\boldsymbol{y})\hat{\psi}(\boldsymbol{x}). \tag{6.62}$$

The ensuing Heisenberg equation

$$i\hbar\frac{\partial\hat{\psi}}{\partial t}(\boldsymbol{x},t) = [\hat{\psi}(\boldsymbol{x},t),\hat{H}(t)] = H_{\boldsymbol{x}}\hat{\psi}(\boldsymbol{x},t) + \int d^3y\,\hat{\psi}^\dagger(\boldsymbol{y},t)V_I(\boldsymbol{x},\boldsymbol{y})\hat{\psi}(\boldsymbol{y},t)\,\hat{\psi}(\boldsymbol{x},t). \tag{6.63}$$

is of the same form as the classical field equation (6.100) in Exercise 25 (for both bosons and fermions). Using the properties

$$\langle 0|\,\hat{\psi}^\dagger(\boldsymbol{x},t) = 0 \quad \text{and} \quad [\hat{\psi}(\boldsymbol{x},t),\hat{\psi}^\dagger(\boldsymbol{y},t)\hat{\psi}(\boldsymbol{y},t)] = \delta(\boldsymbol{x}-\boldsymbol{y})\hat{\psi}(\boldsymbol{y},t) \tag{6.64}$$

one can show that even with interparticle interactions included, the equation (6.63) for quantum field implies the N-particle Schrödinger equation (6.59):

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} &= \sum_{p=1}^{N} H_{\boldsymbol{x}_p}\Psi + \frac{1}{\sqrt{N!}}\langle 0|\sum_{p=1}^{N}\hat{\psi}(\boldsymbol{x}_1,t)\ldots\int d^3y\,\hat{\psi}^\dagger(\boldsymbol{y},t)V_I(\boldsymbol{x}_p,\boldsymbol{y})\hat{\psi}(\boldsymbol{y},t)\,\hat{\psi}(\boldsymbol{x}_p,t)\ldots\hat{\psi}(\boldsymbol{x}_N,t)\,|\Psi\rangle \\ &= \sum_{p=1}^{N} H_{\boldsymbol{x}_p}\Psi + \sum_{p'<p} V_I(\boldsymbol{x}_p,\boldsymbol{x}_{p'})\Psi, \end{aligned} \tag{6.65}$$

where, as in the case of free bosons or fermions,

$$\Psi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N,t)=\frac{1}{\sqrt{N!}}\langle 0|\,\hat{\psi}(\boldsymbol{x}_1,t)\ldots\hat{\psi}(\boldsymbol{x}_N,t)\,|\Psi\rangle\,. \tag{6.66}$$

Eq. (6.63) and its Hermitian conjugate form a coupled system of non-linear partial integro-differential equations for operator-valued functions $\hat{\psi}$ and $\hat{\psi}^\dagger$. A simple mode expansion

$$\hat{\psi}(\boldsymbol{x},t)=\sum_k \hat{a}_k(t)u_k(\boldsymbol{x})\quad,\quad \hat{a}_k(t)=e^{-\frac{i}{\hbar}t\,\epsilon_k}\hat{a}_k \tag{6.67}$$

will not yield an exact solution. Nevertheless, it can serve as a reasonably good approximative solution, provided that the modes u_k and its corresponding energies ϵ_k properly take into account the interacting part of the Hamiltonian (6.62).

For simplicity, let us focus on contact interactions, which reduce the non-local Hamiltonian (6.62) to a local one,

$$V_I(\boldsymbol{x},\boldsymbol{y})=g\,\delta(\boldsymbol{x}-\boldsymbol{y})\quad\rightarrow\quad \hat{H}=\int d^3x\,\hat{\psi}^\dagger(\boldsymbol{x})H_{\boldsymbol{x}}\hat{\psi}(\boldsymbol{x})+\frac{g}{2}\int d^3x\,\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}(\boldsymbol{x})\hat{\psi}(\boldsymbol{x}), \tag{6.68}$$

and so our interacting system is described by a local field theory with Lagrangian density

$$\mathscr{L}=i\hbar\psi^*\partial_t\psi-\frac{\hbar^2}{2m}(\partial_i\psi^*)(\partial_i\psi)-V_0(\boldsymbol{x})\psi^*\psi-\frac{g}{2}(\psi^*\psi)^2. \tag{6.69}$$

We will assume that the particles are bosons, and expand the quantum field $\hat{\psi}(\boldsymbol{x})$ (as in Eq. (6.33)) into an orthonormal set of functions $u_0,u_1,\ldots$:

$$\hat{\psi}(\boldsymbol{x})=\hat{a}_0u_0(\boldsymbol{x})+\sum_{k=1}^{\infty}\hat{a}_ku_k(\boldsymbol{x}),\quad\text{where}\quad[\hat{a}_k,\hat{a}^\dagger_{k'}]=\delta_{kk'}\quad(k,k'=0,1,\ldots). \tag{6.70}$$

To select an optimal *one-particle ground-state wave-function* $u_0(\boldsymbol{x})$ we consider the N-particle state

$$|\Psi_N\rangle=\frac{(\hat{a}_0^\dagger)^N}{\sqrt{N!}}\,|0\rangle\,,\quad\text{or } N\text{-particle wave-function}\quad \Psi_N(\boldsymbol{x}_1,\ldots\boldsymbol{x}_N)=u_0(\boldsymbol{x}_1)\ldots u_0(\boldsymbol{x}_N), \tag{6.71}$$

and minimize its energy expectation value. (This product state will *not* be a true lowest-energy N-particle state, but it will represent a useful approximation thereof.) Since

$$\hat{\psi}(\boldsymbol{x})\,|\Psi_N\rangle=u_0(\boldsymbol{x})\,\hat{a}_0\,|\Psi_N\rangle\quad,\quad\langle\Psi_N|\,\hat{\psi}^\dagger(\boldsymbol{x})=u_0^*(\boldsymbol{x})\,\langle\Psi_N|\,\hat{a}_0^\dagger, \tag{6.72}$$

we have

$$\begin{aligned}\langle\Psi_N|\,\hat{H}\,|\Psi_N\rangle&=\int d^3x\,u_0^*H_{\boldsymbol{x}}u_0\,\langle\Psi_N|\,\hat{a}_0^\dagger\hat{a}_0\,|\Psi_N\rangle+\frac{g}{2}\int d^3x\,(u_0^*u_0)^2\,\langle\Psi_N|\,\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0\,|\Psi_N\rangle\\&=\int d^3x\,u_0^*H_{\boldsymbol{x}}u_0N+\frac{g}{2}\int d^3x\,(u_0^*u_0)^2N(N-1),\end{aligned} \tag{6.73}$$

where we have used the fact that $\hat{a}_0^\dagger\hat{a}_0=\hat{n}_0$ is the number operators of mode $k=0$, and $\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0=\hat{a}_0^\dagger(\hat{a}_0\hat{a}_0^\dagger-1)\hat{a}_0=\hat{n}_0^2-\hat{n}_0$.

We consider a large number of particles $N \gg 1$, and define the *condensate wave-function* $\psi(\boldsymbol{x}) = \sqrt{N} u_0(\boldsymbol{x})$, in terms of which the energy expectation value becomes functional of ψ and ψ^* (treated as independent fields)

$$E_N[\psi, \psi^*] \equiv \langle \Psi_N | \hat{H} | \Psi_N \rangle = \int d^3x \Big(\psi^* H_{\boldsymbol{x}} \psi + \frac{g}{2} (\psi^* \psi)^2 \Big), \tag{6.74}$$

subject to the normalization constraint $\int d^3x \, |\psi(\boldsymbol{x})|^2 = N$. (It is worth to note that the expression for E_N coincides with the expression (6.68) for the Hamiltonian operator $\hat{H}$ if we formally replace the quantum field $\hat{\psi}(\boldsymbol{x})$ with the classical field $\psi(\boldsymbol{x})$.) The energy is minimal provided that

$$\frac{\delta\big(E_N - \mu(\int d^3x \, \psi^* \psi - N)\big)}{\delta \psi^*(\boldsymbol{x})} = H_{\boldsymbol{x}} \psi(\boldsymbol{x}) + g \psi^*(\boldsymbol{x}) \psi^2(\boldsymbol{x}) - \mu \psi(\boldsymbol{x}) = 0, \tag{6.75}$$

that is, if $\psi(\boldsymbol{x})$ satisfies the so-called *Gross-Pitaevskii equation*

$$\left(-\frac{\hbar^2}{2m} \Delta_{\boldsymbol{x}} + V_0(\boldsymbol{x}) + g |\psi(\boldsymbol{x})|^2 \right) \psi(\boldsymbol{x}) = \mu \psi(\boldsymbol{x}), \tag{6.76}$$

where the Lagrange multiplier μ is determined from the normalization constraint.

This equation is used in practice to describe systems of bosons at low energies (Bose-Einstein condensates). It resembles the time-independent Schrödinger equation with an additional non-linear term $g|\psi|^2\psi$, which accounts for the interparticle (contact) interactions. In Exercise 27 we will find explicit solutions of Eq. (6.76) in one spatial dimension for $V_0 = 0$ and $g < 0$ (attractive interparticle interactions).

6.3 Exercises

Exercise 23. *Vacuum correlations.* For any $n, n' = 1, \ldots, N$ calculate

$$\langle 0|\,\hat{q}_n\hat{q}_{n'}\,|0\rangle \quad \text{and} \quad \langle 0|\,\hat{q}_n\,|0\rangle\,, \qquad \text{given} \qquad \langle\vec{q}|0\rangle = \psi_0(\vec{q}) = \mathcal{N}\exp\left(-\frac{1}{2\hbar}\vec{q}^{\,T}\mathsf{A}\,\vec{q}\right), \tag{6.77}$$

where $\mathcal{N}$ is a normalization factor such that $\langle 0|0\rangle = 1$, and the matrix A is symmetric positive-definite.

Solution:

In order to calculate the integral

$$\langle 0|\,\hat{q}_n\hat{q}_{n'}\,|0\rangle = \int d^N q\,\langle 0|\vec{q}\rangle\,\langle\vec{q}|\,\hat{q}_n\hat{q}_{n'}\,|0\rangle = \int d^N q\, q_n q_{n'}|\psi_0(\vec{q})|^2 = \mathcal{N}^2\int d^N q\, q_n q_{n'}\exp\left(-\frac{1}{\hbar}\vec{q}^{\,T}\mathsf{A}\,\vec{q}\right) \tag{6.78}$$

we employ the method of generating functions to write

$$\langle 0|\,\hat{q}_n\hat{q}_{n'}\,|0\rangle = \hbar^2\frac{\partial}{\partial j_n}\frac{\partial}{\partial j_{n'}}\bigg|_{\vec{j}=\vec{0}}\langle 0|\,e^{\frac{1}{\hbar}j_m\hat{q}_m}\,|0\rangle\,, \tag{6.79}$$

where

$$\langle 0|\,e^{\frac{1}{\hbar}j_m\hat{q}_m}\,|0\rangle = \mathcal{N}^2\int d^N q\,\exp\left(-\frac{1}{\hbar}\vec{q}^{\,T}\mathsf{A}\,\vec{q} + \frac{1}{\hbar}\vec{j}^{\,T}\vec{q}\right), \tag{6.80}$$

and $\vec{j} = (j_m)_{m=1}^N$ is a vector of auxiliary variables. Now complete the square in the exponential,

$$-\frac{1}{\hbar}\vec{q}^{\,T}\mathsf{A}\,\vec{q} + \frac{1}{\hbar}\vec{j}^{\,T}\vec{q} = -\frac{1}{\hbar}\left(\vec{q} - \frac{1}{2}\mathsf{A}^{-1}\vec{j}\right)^T\mathsf{A}\left(\vec{q} - \frac{1}{2}\mathsf{A}^{-1}\vec{j}\right) + \frac{1}{4\hbar}\vec{j}^{\,T}\mathsf{A}^{-1}\vec{j} \tag{6.81}$$

(using the fact that $\mathsf{A}^T = \mathsf{A} \Rightarrow (\mathsf{A}^{-1})^T = \mathsf{A}^{-1}$), and shift the integration variables to arrive at

$$\langle 0|\,e^{\frac{1}{\hbar}\vec{j}^{\,T}\hat{\vec{q}}}\,|0\rangle = \langle 0|0\rangle\exp\left(\frac{1}{4\hbar}\vec{j}^{\,T}\mathsf{A}^{-1}\vec{j}\right) = \exp\left(\frac{1}{4\hbar}\vec{j}^{\,T}\mathsf{A}^{-1}\vec{j}\right). \tag{6.82}$$

From this generating function we obtain the sought-for vacuum expectation values as follows:

$$\begin{aligned}
\langle 0|\,\hat{q}_n\hat{q}_{n'}\,|0\rangle &= \hbar^2\frac{\partial}{\partial j_n}\frac{\partial}{\partial j_{n'}}\bigg|_{\vec{j}=\vec{0}}\left(1 + \frac{1}{4\hbar}\vec{j}^{\,T}\mathsf{A}^{-1}\vec{j} + \ldots\right) \\
&= \frac{\hbar}{4}\frac{\partial}{\partial j_n}\frac{\partial}{\partial j_{n'}}\sum_{m,m'}(\mathsf{A}^{-1})_{mm'}j_m j_{m'} \\
&= \frac{\hbar}{4}\sum_{m,m'}(\mathsf{A}^{-1})_{mm'}(\delta_{mn}\delta_{m'n'} + \delta_{mn'}\delta_{m'n}) \\
&= \frac{\hbar}{4}\left((\mathsf{A}^{-1})_{nn'} + (\mathsf{A}^{-1})_{n'n}\right) \\
&= \frac{\hbar}{2}(\mathsf{A}^{-1})_{nn'},
\end{aligned} \tag{6.83}$$

and

$$\langle 0|\,\hat{q}_n\,|0\rangle = \hbar\frac{\partial}{\partial j_n}\bigg|_{\vec{j}=\vec{0}}\left(1 + \frac{1}{4\hbar}\vec{j}^{\,T}\mathsf{A}^{-1}\vec{j} + \ldots\right) = 0, \tag{6.84}$$

since there is no term linear in $\vec{j}$ in the expansion of $\langle 0|\,e^{\frac{1}{\hbar}\vec{j}^{\,T}\hat{\vec{q}}}\,|0\rangle$.

Exercise 24. *Two coupled quantum oscillators.* Consider a quantum system of two coupled harmonic oscillators described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}_1^2}{2M} + \frac{\hat{p}_2^2}{2M} + \frac{\kappa_0}{2}\hat{q}_1^2 + \frac{\kappa_0}{2}\hat{q}_2^2 + \frac{\kappa_I}{2}(\hat{q}_2 - \hat{q}_1)^2, \qquad \kappa_0, \kappa_I > 0. \tag{6.85}$$

1. Using the mode decomposition from Exercise 19 write the Hamiltonian in terms of modes' creation and annihilation operators.
2. Write down the ground state wave-function.
3. Calculate the vacuum correlation $\langle 0|\, \hat{q}_1\hat{q}_2\, |0\rangle$, and find its behaviour for small or large interaction constant κ_I.

Solution:

1. Recall that

$$\mathsf{U} = \begin{pmatrix} \kappa_0 + \kappa_I & -\kappa_I \\ -\kappa_I & \kappa_0 + \kappa_I \end{pmatrix} = \mathsf{V}\,\mathsf{K}\,\mathsf{V}^T, \quad \mathsf{K} = M\Omega^2 = \begin{pmatrix} \kappa_0 & 0 \\ 0 & \kappa_0 + 2\kappa_I \end{pmatrix}, \quad \mathsf{V} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{6.86}$$

Then, according to Eq. (6.14),

$$\hat{H} = \hbar\omega_1 \left(\hat{a}_1^\dagger \hat{a}_1 + \frac{1}{2}\right) + \hbar\omega_2 \left(\hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2}\right), \quad \text{where} \quad \omega_1 = \sqrt{\frac{\kappa_0}{M}}, \quad \omega_2 = \sqrt{\frac{\kappa_0 + 2\kappa_I}{M}}. \tag{6.87}$$

2. By Eqs. (6.19) and (6.20),

$$\psi_0(\vec{q}) = \mathcal{N} \exp\left(-\frac{1}{2\hbar}\vec{q}^{\,T}\mathsf{A}\,\vec{q}\right), \tag{6.88}$$

with

$$\mathsf{A} = M\mathsf{V}\Omega\mathsf{V}^T = \frac{M}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{M}{2}\begin{pmatrix} \omega_1 + \omega_2 & \omega_1 - \omega_2 \\ \omega_1 - \omega_2 & \omega_1 + \omega_2 \end{pmatrix}. \tag{6.89}$$

3. According to the formula (6.21) we need to calculate the inverse of A, $\mathsf{A}^{-1} = \mathsf{V}(M\Omega)^{-1}\mathsf{V}^T$. Explicitly, we find

$$\mathsf{A}^{-1} = \frac{1}{2M}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\omega_1} & 0 \\ 0 & \frac{1}{\omega_2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2M\omega_1\omega_2}\begin{pmatrix} \omega_1 + \omega_2 & \omega_2 - \omega_1 \\ \omega_2 - \omega_1 & \omega_1 + \omega_2 \end{pmatrix}. \tag{6.90}$$

The vacuum correlation between the two degrees of freedom is then given by

$$\langle 0|\, \hat{q}_1\hat{q}_2\, |0\rangle = \frac{\hbar}{2}(\mathsf{A}^{-1})_{1,2} = \frac{\hbar}{4M}\frac{\omega_2 - \omega_1}{\omega_1\omega_2} = \frac{\hbar}{4\sqrt{M\kappa_0}}\frac{\sqrt{\kappa_0 + 2\kappa_I} - \sqrt{\kappa_0}}{\sqrt{\kappa_0 + 2\kappa_I}}, \tag{6.91}$$

which can be approximated for $\kappa_I \to 0$ as

$$\langle 0|\, \hat{q}_1\hat{q}_2\, |0\rangle = \frac{\hbar}{4\sqrt{M\kappa_0}}\frac{\sqrt{1 + \frac{2\kappa_I}{\kappa_0}} - 1}{\sqrt{1 + \frac{2\kappa_I}{\kappa_0}}} \approx \frac{\hbar}{4\sqrt{M\kappa_0}}\frac{\kappa_I}{\kappa_0}, \tag{6.92}$$

and for $\kappa_I \to +\infty$ as

$$\langle 0|\, \hat{q}_1\hat{q}_2\, |0\rangle \approx \frac{\hbar}{4\sqrt{M\kappa_0}}. \tag{6.93}$$

Exercise 25. *Free and interacting Schrödinger field.* Consider a complex one-component field $\psi(x)$ with action functional $S = S_0 + S_I$, where

$$S_0[\psi, \psi^*] = \int d^4x \, \mathscr{L}_0 \quad , \quad \mathscr{L}_0 = i\hbar\psi^*\partial_t\psi - \frac{\hbar^2}{2m}(\partial_i\psi^*)(\partial_i\psi) - V_0(\boldsymbol{x})\psi^*\psi \tag{6.94}$$

is the 'free' action and 'free' Lagrangian (they don't contain interparticle interactions), and

$$S_I[\psi, \psi^*] = -\frac{1}{2}\int dt\, d^3x\, d^3y\, \psi^*(\boldsymbol{x}, t)\psi^*(\boldsymbol{y}, t)V_I(\boldsymbol{x}, \boldsymbol{y})\psi(\boldsymbol{y}, t)\psi(\boldsymbol{x}, t) \quad , \quad V_I(\boldsymbol{x}, \boldsymbol{y}) = V_I(\boldsymbol{y}, \boldsymbol{x}), \tag{6.95}$$

is the 'interacting part' of the action.

1. Derive the Euler-Lagrange equations for the free action S_0.
2. Derive the Euler-Lagrange equations for the full action S. (Mind its non-locality.)

Solution:

Note that since the action is real (up to a boundary term),

$$S^* - S = \int d^4x \, (-i\hbar\psi\partial_t\psi^* - i\hbar\psi^*\partial_t\psi) = -i\hbar \int d^4x \, \partial_t(\psi\psi^*), \tag{6.96}$$

the Euler-Lagrange equation for variation with respect to ψ is complex conjugate to that for ψ^*:

$$\frac{\delta S}{\delta\psi} = \left(\frac{\delta S^*}{\delta\psi^*}\right)^* = \left(\frac{\delta S}{\delta\psi^*}\right)^* = 0. \tag{6.97}$$

Hence, we only need to calculate the variation $\frac{\delta S}{\delta\psi^*}$.

1. For the free action S_0 we find an ordinary Schrödinger equation with potential V_0:
$$\begin{aligned}\frac{\delta S_0}{\delta\psi^*} &= \frac{\partial\mathscr{L}_0}{\partial\psi^*} - \partial_t\frac{\partial\mathscr{L}_0}{\partial(\partial_t\psi^*)} - \partial_i\frac{\partial\mathscr{L}_0}{\partial(\partial_i\psi^*)} \\ &= i\hbar\partial_t\psi - V_0(\boldsymbol{x})\psi - \partial_t(0) - \partial_i\Big(-\frac{\hbar^2}{2m}\partial_i\psi\Big) \\ &= i\hbar\partial_t\psi - \Big(-\frac{\hbar^2}{2m}\Delta + V_0(\boldsymbol{x})\Big)\psi = 0\end{aligned} \tag{6.98}$$

2. Denote $x = (\boldsymbol{x}, t)$, $x' = (\boldsymbol{x}', t')$, $y' = (\boldsymbol{y}', t')$, and calculate
$$\begin{aligned}\frac{\delta S_I}{\delta\psi^*(x)} &= -\frac{1}{2}\int dt'\, d^3x'\, d^3y'\, \frac{\delta\big(\psi^*(x')\psi^*(y')\big)}{\delta\psi^*(x)} V_I(\boldsymbol{x}', \boldsymbol{y}')\psi(y')\psi(x') \\ &= -\frac{1}{2}\int dt'\, d^3x'\, d^3y'\, \Big(\delta(x'-x)\psi^*(y') + \psi^*(x')\delta(y'-x)\Big)V_I(\boldsymbol{x}', \boldsymbol{y}')\psi(y')\psi(x') \\ &= -\int d^3y\, \psi^*(\boldsymbol{y}, t)V_I(\boldsymbol{x}, \boldsymbol{y})\psi(\boldsymbol{y}, t)\psi(\boldsymbol{x}, t),\end{aligned} \tag{6.99}$$
where in the last step we used the assumption of symmetry of V_I in its arguments. Hence, the full action yields the equation of motion
$$\frac{\delta(S_0 + S_I)}{\delta\psi^*} = 0 \quad \to \quad i\hbar\frac{\partial\psi}{\partial t} = \Big(-\frac{\hbar^2}{2m}\Delta + V_0(\boldsymbol{x}) + \int d^3y\, \psi^*(\boldsymbol{y}, t)V_I(\boldsymbol{x}, \boldsymbol{y})\psi(\boldsymbol{y}, t)\Big)\psi(\boldsymbol{x}, t). \tag{6.100}$$
This equation is non-local — the interaction part represents the energy at point $\boldsymbol{x}$ due to the field (the 'density' $\psi^*\psi$) integrated over the whole space.

Exercise 26. *Multi-particle wave-functions.* From the expansion

$$\hat{\psi}(\boldsymbol{x}) = \sum_k \hat{a}_k u_k(\boldsymbol{x}) \quad \text{for bosons,} \quad \text{and} \quad \hat{\psi}(\boldsymbol{x}) = \sum_k \hat{b}_k u_k(\boldsymbol{x}) \quad \text{for fermions,} \tag{6.101}$$

of the field operator in terms of an orthonormal set of functions, show that:

1. For bosonic fields

$$\langle 0|\, \hat{\psi}(\boldsymbol{x}_1) \ldots \hat{\psi}(\boldsymbol{x}_N)\, \hat{a}^\dagger_{k_1} \ldots \hat{a}^\dagger_{k_N} \,|0\rangle = \sum_{\sigma \in \mathcal{S}_N} u_{k_1}(\boldsymbol{x}_{\sigma(1)}) \ldots u_{k_N}(\boldsymbol{x}_{\sigma(N)}). \tag{6.102}$$

2. For fermionic fields

$$\langle 0|\, \hat{\psi}(\boldsymbol{x}_1) \ldots \hat{\psi}(\boldsymbol{x}_N)\, \hat{b}^\dagger_{k_N} \ldots \hat{b}^\dagger_{k_1} \,|0\rangle = \sum_{\sigma \in \mathcal{S}_N} (\operatorname{sgn} \sigma)\, u_{k_1}(\boldsymbol{x}_{\sigma(1)}) \ldots u_{k_N}(\boldsymbol{x}_{\sigma(N)}). \tag{6.103}$$

Solution:

1. We introduce auxiliary variables $\alpha_k \in \mathbb{R}$ (one for each mode), and consider the generating series

$$\langle 0|\, \hat{\psi}(\boldsymbol{x}_1) \ldots \hat{\psi}(\boldsymbol{x}_N)\, e^{\sum_k \alpha_k \hat{a}^\dagger_k} \,|0\rangle = \frac{1}{N!} \sum_{k_1,\ldots,k_N} \alpha_{k_1} \ldots \alpha_{k_N} \,\langle 0|\, \hat{\psi}(\boldsymbol{x}_1) \ldots \hat{\psi}(\boldsymbol{x}_N)\, \hat{a}^\dagger_{k_1} \ldots \hat{a}^\dagger_{k_N} \,|0\rangle\,, \tag{6.104}$$

where all other terms in the expansion of the exponential (i.e., those with the number of creation operators different from N) vanish under $\langle 0| \ldots |0\rangle$.

Another way of writing the left-hand side makes use of the Campbell identity (2.14)

$$\exp\left(-\sum_k \alpha_k \hat{a}^\dagger_k\right) \hat{\psi}(\boldsymbol{x}) \exp\left(\sum_k \alpha_k \hat{a}^\dagger_k\right) = \hat{\psi}(\boldsymbol{x}) + \left[\hat{\psi}(\boldsymbol{x}), \sum_k \alpha_k \hat{a}^\dagger_k\right] = \hat{\psi}(\boldsymbol{x}) + \sum_k \alpha_k u_k(\boldsymbol{x}), \tag{6.105}$$

where we have used the fact that $[\hat{\psi}(\boldsymbol{x}), \hat{a}^\dagger_k] = u_k(\boldsymbol{x})$ (which commutes with everything, so higher commutators vanish). Since $\langle 0|\, \hat{a}^\dagger_k = 0$ and $\hat{\psi}(\boldsymbol{x}) \,|0\rangle = 0$, we can write

$$\begin{aligned}
&\langle 0|\, \hat{\psi}(\boldsymbol{x}_1) \ldots \hat{\psi}(\boldsymbol{x}_N)\, e^{\sum_k \alpha_k \hat{a}^\dagger_k} \,|0\rangle \\
&= \langle 0|\, e^{-\sum_k \alpha_k \hat{a}^\dagger_k} \hat{\psi}(\boldsymbol{x}_1) e^{\sum_k \alpha_k \hat{a}^\dagger_k} \ldots e^{-\sum_k \alpha_k \hat{a}^\dagger_k} \hat{\psi}(\boldsymbol{x}_N)\, e^{\sum_k \alpha_k \hat{a}^\dagger_k} \,|0\rangle \\
&= \langle 0| \left(\hat{\psi}(\boldsymbol{x}_1) + \sum_{k_1} \alpha_{k_1} u_{k_1}(\boldsymbol{x}_1)\right) \ldots \left(\hat{\psi}(\boldsymbol{x}_N) + \sum_{k_N} \alpha_{k_N} u_{k_N}(\boldsymbol{x}_N)\right) |0\rangle \\
&= \langle 0| \sum_{k_1} \alpha_{k_1} u_{k_1}(\boldsymbol{x}_1) \ldots \sum_{k_N} \alpha_{k_N} u_{k_N}(\boldsymbol{x}_N) \,|0\rangle \\
&= \sum_{k_1,\ldots,k_N} \alpha_{k_1} \ldots \alpha_{k_N}\, u_{k_1}(\boldsymbol{x}_1) \ldots u_{k_N}(\boldsymbol{x}_N) \\
&= \sum_{k_1,\ldots,k_N} \alpha_{k_1} \ldots \alpha_{k_N} \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} u_{k_1}(\boldsymbol{x}_{\sigma(1)}) \ldots u_{k_N}(\boldsymbol{x}_{\sigma(N)}).
\end{aligned} \tag{6.106}$$

Since the series coefficients of expansions (6.104) and (6.106) must be equal (at every monomial $\alpha_{k_1} \ldots \alpha_{k_N}$) we obtain Eq. (6.102).

2. The fermionic case is more subtle, although the general idea is the same as in the case of bosons. To successfully implement the Campbell identity, the commutator

$$\Big[\hat{\psi}(\boldsymbol{x}), \sum_k \beta_k \hat{b}_k^\dagger\Big] = \sum_{k,\ell} u_\ell(\boldsymbol{x}) \Big[\hat{b}_\ell, \beta_k \hat{b}_k^\dagger\Big] = \sum_{k,\ell} u_\ell(\boldsymbol{x}) \Big(\hat{b}_\ell \beta_k \hat{b}_k^\dagger - \beta_k \hat{b}_k^\dagger \hat{b}_\ell\Big) \tag{6.107}$$

needs to reduce to a simple expression using the anticommutation rule $\{b_\ell, b_k^\dagger\} = \delta_{\ell k}$. To this end we postulate that the β_k are not ordinary complex numbers, but rather the so-called *Grassmann symbols* (also known as *Grassmann variables*, *Grassmann numbers*) that obey the anticommutation rules (the Grassmann algebra)

$$\{\beta_k, \beta_\ell\} = 0 \quad \text{and} \quad \{\beta_k, \hat{b}_\ell\} = \{\beta_k, \hat{b}_\ell^\dagger\} = 0 \quad (\forall k, \ell). \tag{6.108}$$

(The modes $u_k(\boldsymbol{x})$ remain ordinary complex-valued functions that commute with everything.)

With these definitions we obtain

$$\Big[\hat{\psi}(\boldsymbol{x}), \sum_k \beta_k \hat{b}_k^\dagger\Big] = -\sum_{k,\ell} u_\ell(\boldsymbol{x}) \beta_k \{\hat{b}_\ell, \hat{b}_k^\dagger\} = -\sum_k u_k(\boldsymbol{x}) \beta_k \quad , \quad [\beta_\ell, \beta_k \hat{b}_k^\dagger] = 0, \tag{6.109}$$

and so

$$\exp\left(-\sum_k \beta_k \hat{b}_k^\dagger\right) \hat{\psi}(\boldsymbol{x}) \exp\left(\sum_k \beta_k \hat{b}_k^\dagger\right) = \hat{\psi}(\boldsymbol{x}) + \Big[\hat{\psi}(\boldsymbol{x}), \sum_k \beta_k \hat{b}_k^\dagger\Big] = \hat{\psi}(\boldsymbol{x}) - \sum_k u_k(\boldsymbol{x}) \beta_k. \tag{6.110}$$

Hence, similarly as in (6.106) we obtain

$$\langle 0|\, \hat{\psi}(\boldsymbol{x}_1) \dots \hat{\psi}(\boldsymbol{x}_N)\, e^{\sum_k \beta_k \hat{b}_k^\dagger} |0\rangle = (-1)^N \sum_{k_1,\dots,k_N} u_{k_1}(\boldsymbol{x}_1) \dots u_{k_N}(\boldsymbol{x}_N)\, \beta_{k_1} \dots \beta_{k_N}, \tag{6.111}$$

while Eq. (6.104) has a fermionic analogue

$$\begin{aligned}
\langle 0|\, \hat{\psi}(\boldsymbol{x}_1) \dots \hat{\psi}(\boldsymbol{x}_N)\, e^{\sum_k \beta_k \hat{b}_k^\dagger} |0\rangle &= \frac{1}{N!} \sum_{k_1,\dots,k_N} \langle 0|\, \hat{\psi}(\boldsymbol{x}_1) \dots \hat{\psi}(\boldsymbol{x}_N)\, \beta_{k_N} \hat{b}_{k_N}^\dagger \dots \beta_{k_1} \hat{b}_{k_1}^\dagger |0\rangle \\
&= \frac{-1}{N!} \sum_{k_1,\dots,k_N} \langle 0|\, \hat{\psi}(\boldsymbol{x}_1) \dots \hat{\psi}(\boldsymbol{x}_N)\, \hat{b}_{k_N}^\dagger \dots \beta_{k_1} \hat{b}_{k_1}^\dagger |0\rangle\, \beta_{k_N} \\
&= \frac{(-1)^N}{N!} \sum_{k_1,\dots,k_N} \langle 0|\, \hat{\psi}(\boldsymbol{x}_1) \dots \hat{\psi}(\boldsymbol{x}_N)\, \hat{b}_{k_N}^\dagger \dots \hat{b}_{k_1}^\dagger |0\rangle\, \beta_{k_1} \dots \beta_{k_N}.
\end{aligned} \tag{6.112}$$

Comparing now (after *antisymmetrization* — due to anticommuting nature of β_k's) a particular term $\beta_{k_1} \dots \beta_{k_N}$ of the two expansions yields Eq. (6.103).

However strange they might seem, the Grassmann symbols are frequently used when dealing with quantum field theory for fermions, especially in the path integral formulation.

Exercise 27. *Gross-Pitaevskii equation in 1D.* Find real solutions $\psi(x)$ of the differential equation

$$-\frac{\hbar^2}{2m}\psi'' + g\psi^3 = \mu\psi, \quad \text{where} \quad g < 0, \tag{6.113}$$

and determine μ so that

$$\int dx\, \psi^2(x) = N. \tag{6.114}$$

Solution:

Casting Eq. (6.113) in the form

$$\psi'' = -\frac{2m}{\hbar^2}(\mu\psi - g\psi^3) = -\frac{\partial}{\partial\psi}\Big(\frac{\mu}{2}\psi^2 - \frac{g}{4}\psi^4\Big)\frac{2m}{\hbar^2}, \tag{6.115}$$

which is analogous to the Newton's law of classical mechanics $\ddot{x} = -\partial_x V$ (here $\psi(x) \leftrightarrow x(t)$), we can immediately write, in analogy with the law of conservation of total mechanical energy,

$$\frac{1}{2}(\psi')^2 + \frac{2m}{\hbar^2}\Big(\frac{\mu}{2}\psi^2 - \frac{g}{4}\psi^4\Big) = C. \tag{6.116}$$

In order to satisfy Eq. (6.114), ψ and ψ' tend to 0 as $x \to \pm\infty$, and hence $C = 0$. Moreover, since $g < 0$, we must have $\mu < 0$, and a multiplication by $\frac{2}{\psi^4}$ yields

$$\Big(\frac{\psi'}{\psi^2}\Big)^2 + \frac{2m}{\hbar^2}\Big(\frac{\mu}{\psi^2} - \frac{g}{2}\Big) = (f')^2 - \frac{2m}{\hbar^2}|\mu|\, f^2 + \frac{2m}{\hbar^2}\frac{|g|}{2} = 0, \qquad \text{where} \qquad f \equiv \frac{1}{\psi}. \tag{6.117}$$

This first-order differential equation for $f(x)$ can be solved by separation of variables. However, it is faster to recall the identity $\sinh^2 x - \cosh^2 x + 1 = 0$, and look for the solution in the form

$$f(x) = \alpha\cosh\big(\beta(x - x_0)\big), \quad \text{where} \quad \alpha, \beta > 0 \quad \text{and} \quad x_0 \in \mathbb{R}. \tag{6.118}$$

The constants α and β are determined by plugging this ansatz into Eq. (6.117),

$$\beta = \sqrt{\frac{2m}{\hbar^2}|\mu|} \quad , \quad \alpha = \sqrt{\frac{|g|}{2|\mu|}}, \tag{6.119}$$

while x_0 remains as an arbitrary integration constant.

To eliminate the Lagrange multiplier μ we evaluate the integral

$$\int_{-\infty}^{+\infty} dx\, \psi^2(x) = \int_{-\infty}^{+\infty} dx\, \frac{1}{\alpha^2\cosh^2(\beta x)} = \frac{1}{\alpha^2\beta}\big[\tanh(\beta x)\big]_{-\infty}^{+\infty} = \frac{2}{\alpha^2\beta} = \frac{4}{|g|}\sqrt{\frac{\hbar^2}{2m}}\sqrt{|\mu|} = N. \tag{6.120}$$

Altogether, we obtain the condensate wave-function (a 'bright soliton')

$$\psi(x) = \frac{\psi(x_0)}{\cosh(\frac{x-x_0}{L})} \quad , \quad L = \frac{1}{\beta} = \frac{\hbar}{\sqrt{2m|\mu|}} = \frac{\hbar^2}{2m}\frac{4}{N|g|}, \tag{6.121}$$

which represents a cloud of bosons centred around point x_0, and with the size is of the order L.

Remarks:

The arbitrariness of the point x_0 reflects the symmetry of the system (namely, of the Lagrangian (6.69) with $V_0 = 0$) under spatial translations. Once the location of the cloud x_0 is decided, this symmetry is *spontaneously broken* (the cloud itself is not translationally invariant). The resulting location, however, cannot be deduced from the Lagrangian of the system (hence the adjective 'spontaneous'). In practice the Bose-Einstein condensates are trapped in an external potential $V_0(x)$, e.g., $\frac{\omega}{2}(x-x_0)^2$, which decides the location of the cloud. The explicit dependence of V_0 (and hence the Lagrangian) on x then breaks the translational symmetry *explicitly.*

Chapter 7

Canonical quantization of Klein-Gordon field

Let us turn back to natural units $\hbar = c = 1$.

The simplest model of relativistic field theory is a real scalar one-component field described by the Klein-Gordon Lagrangian (density)

$$\mathscr{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2. \tag{7.1}$$

This falls within the category of 'free' field theories, whose Lagrangian is at most quadratic in fields and their derivatives, and hence leads to linear equations of motion. (We will now be dealing with free fields for some time — until Chapter 11.) The corresponding Hamiltonian is given (according to Exercise 21) by

$$\pi(\boldsymbol{x}) = \partial_t \phi(\boldsymbol{x}) \quad \rightarrow \quad H = \int d^3x \, \frac{1}{2}\Big(\pi^2 + (\partial_i \phi)(\partial_i \phi) + m^2\phi^2\Big). \tag{7.2}$$

Quantization consists in promoting the classical fields ϕ and π to operators, and postulating the equal-time canonical commutation relations

$$[\hat{\phi}(\boldsymbol{x}, t), \hat{\pi}(\boldsymbol{y}, t)] = i\,\delta(\boldsymbol{x} - \boldsymbol{y}) \quad \text{and} \quad [\hat{\phi}(\boldsymbol{x}, t), \hat{\phi}(\boldsymbol{y}, t)] = [\hat{\pi}(\boldsymbol{x}, t), \hat{\pi}(\boldsymbol{y}, t)] = 0. \tag{7.3}$$

(The use of commutators, rather than anticommutators, is in fact required by special relativity, as will be discussed in Section 9.1.) Real-valuedness of the classical field ϕ translates to hermiticity on the quantum level: $\hat{\phi}^\dagger = \hat{\phi}$. The classical Hamilton's canonical equations of motion derived in Exercise 21 turn into the quantum Heisenberg equations

$$\partial_t \hat{\phi} = \hat{\pi} \quad , \quad \partial_t \hat{\pi} = -m^2\hat{\phi} + \partial_i \partial_i \hat{\phi}, \quad \text{implying} \quad (\partial^\mu \partial_\mu + m^2)\hat{\phi}(x) = 0, \tag{7.4}$$

which is the (real) Klein-Gordon equation for an operator-valued function $\hat{\phi}$.

In this introductory course we study quantum fields by standard methods of quantum mechanics extended to infinite-dimensional systems. The central role is therefore assumed by the canonical variables and the total Hamiltonian (the total energy of the system), which generates time evolution. This approach, referred to as 'canonical quantization', singles out a preferred time coordinate, and is therefore not manifestly Lorentz invariant. (The fields in equal-time commutation relations (7.3) are taken simultaneously with respect to this particular time 'slicing'.)

Alternatively, quantization can be carried out with the Feynman path integral, which sums over all histories in the configuration space, weighted by phase factors $e^{\frac{i}{\hbar}S}$. In the case of local field theory the action $S = \int d^4x\, \mathscr{L}$ features a Lagrangian density $\mathscr{L}$, which for relativistic theories is a scalar under Lorentz transformations. This approach *is* manifestly Lorentz invariant, but requires familiarity with field-theoretic path integrals. It will be explored in the advanced quantum field theory course (KTPA2).

7.1 Mode expansion of Klein-Gordon field

To study quantum theory of the Klein-Gordon field we note that its Hamiltonian (7.2) can be cast in the form

$$H = \int d^3x\, \frac{\pi^2(\boldsymbol{x})}{2} + \frac{1}{2}\int d^3x\, d^3y\, \phi(\boldsymbol{x})U(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{y}), \quad \text{where} \quad U(\boldsymbol{x},\boldsymbol{y}) = (-\Delta_x + m^2)\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{7.5}$$

which resembles the Hamiltonian of a system of coupled oscillators,

$$H(q_n,p_n) = \sum_n \frac{p_n^2}{2M} + \frac{1}{2}\sum_{n,n'} U_{nn'}q_nq_{n'}, \tag{7.6}$$

studied on the classical level in Section 5.1, and on the quantum level in Section 6.1. The (generalized) function of two variables $U(\boldsymbol{x},\boldsymbol{y})$ is a continuum analogue of the 'potential energy matrix' (or 'stiffness matrix') $U_{nn'}$. In a continuum sense, it describes homogeneous nearest-neighbour coupling in a three-dimensional space, where each point hosts one degree of freedom $\phi(\boldsymbol{x})$ (recall Section 5.1.3, where we studied the continuum limit of a linear chain). The mass M of the discrete oscillators does not appear in the Klein-Gordon Hamiltonian (7.5) (it is absorbed in the light velocity $c = 1$). Note that M has nothing to do with the parameter m of the Klein-Gordon theory, which represents the mass of field excitations entering the relativistic dispersion relation $p^\mu p_\mu = m^2$.

Let us determine normal modes of the Klein-Gordon field theory. The eigenvalue problem for the 'continuum matrix' $U(\boldsymbol{x},\boldsymbol{y})$ can be stated in analogy with the more common discrete case:

$$\sum_{n'} U_{nn'}v_{n'} = \lambda\, v_n \quad \rightsquigarrow \quad \int d^3y\, U(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{y}) = (-\Delta_x + m^2)f(\boldsymbol{x}) = \lambda\, f(\boldsymbol{x}). \tag{7.7}$$

This differential equation is clearly solved by the plane waves

$$f_{\boldsymbol{p}}(\boldsymbol{x}) = \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{(2\pi)^{3/2}} \quad , \quad \lambda_{\boldsymbol{p}} = \boldsymbol{p}^2 + m^2 = \omega_{\boldsymbol{p}}^2 \qquad (\boldsymbol{p} \in \mathbb{R}^3). \tag{7.8}$$

(It is convenient to use complex-valued modes in what follows.) It is worth to note that the function of two variables $F(\boldsymbol{x},\boldsymbol{p}) \equiv f_{\boldsymbol{p}}(\boldsymbol{x})$, where $\boldsymbol{p}$ enumerates the various modes, plays the role of the 'matrix of normal modes' $\mathsf{V} = (V_{nk})$ in Eq. (5.8). In analogy with the discrete case we write

$$q_n(t) = \sum_k V_{nk}\eta_k(t) \quad \rightsquigarrow \quad \phi(\boldsymbol{x},t) = \int d^3p\, F(\boldsymbol{x},\boldsymbol{p})\tilde{\phi}(\boldsymbol{p},t) = \int \frac{d^3p}{(2\pi)^{3/2}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\tilde{\phi}(\boldsymbol{p},t). \tag{7.9}$$

where the 'continuum normal coordinates' $\tilde{\phi}(\boldsymbol{p},t)$, as it turns out, stand for the three-dimensional (i.e., spatial) Fourier transform of the Klein-Gordon field $\phi(\boldsymbol{x},t)$. Since $\phi(\boldsymbol{x},t)$ is real-valued, we observe that

$$\int \frac{d^3p}{(2\pi)^{3/2}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\tilde{\phi}(\boldsymbol{p},t) = \int \frac{d^3p}{(2\pi)^{3/2}}\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\tilde{\phi}^*(\boldsymbol{p},t) \quad \rightarrow \quad \tilde{\phi}^*(\boldsymbol{p},t) = \tilde{\phi}(-\boldsymbol{p},t). \tag{7.10}$$

The continuum analogues of orthonormality and completeness relations (5.10) (extended to complex domain) are:

$$\begin{aligned}
\text{orthonormality}: \quad & (\mathsf{V}^\dagger\mathsf{V})_{kk'} = V^*_{nk}V_{nk'} = \delta_{kk'} \\
\rightsquigarrow \quad & \int d^3x\, F^*(\boldsymbol{x},\boldsymbol{p})F(\boldsymbol{x},\boldsymbol{p}') = \int \frac{d^3x}{(2\pi)^3}\, e^{i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{x}} = \delta(\boldsymbol{p}'-\boldsymbol{p}), \\
\text{completeness}: \quad & (\mathsf{V}\mathsf{V}^\dagger)_{nn'} = V_{nk}V^*_{n'k} = \delta_{nn'} \\
\rightsquigarrow \quad & \int d^3p\, F(\boldsymbol{x},\boldsymbol{p})F^*(\boldsymbol{x}',\boldsymbol{p}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')} = \delta(\boldsymbol{x}-\boldsymbol{x}').
\end{aligned} \tag{7.11}$$

Normal coordinates oscillate in time with their respective angular frequencies (as in Eq. (6.7)). In our continuum (and complex) case this means that, on the quantum level,

$$\hat{\tilde{\phi}}(\boldsymbol{p},t) = \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\Big(\hat{a}_{\boldsymbol{p}}\, e^{-i\omega_{\boldsymbol{p}}t} + \hat{a}^\dagger_{-\boldsymbol{p}}\, e^{i\omega_{\boldsymbol{p}}t}\Big), \quad \text{where} \quad \omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2+m^2}, \tag{7.12}$$

and where the minus sign in $\hat{a}^\dagger_{-\boldsymbol{p}}$ ensures the reality constraint $\hat{\tilde{\phi}}^\dagger(\boldsymbol{p},t) = \hat{\tilde{\phi}}(-\boldsymbol{p},t)$. A general solution of the equations of motion (7.4) (i.e., of the Klein-Gordon equation) is then given by (cf. Eq. (6.17))

$$\begin{aligned}
\hat{\phi}(\boldsymbol{x},t) &= \int d^3p\, \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}}\Big(\hat{a}_{\boldsymbol{p}}\, e^{-i\omega_{\boldsymbol{p}}t} + \hat{a}^\dagger_{-\boldsymbol{p}}\, e^{i\omega_{\boldsymbol{p}}t}\Big) \\
&= \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}}\Big(\hat{a}_{\boldsymbol{p}}\, e^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{-\boldsymbol{p}}\, e^{i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}}\Big).
\end{aligned} \tag{7.13}$$

The substitution $\boldsymbol{p} \rightarrow -\boldsymbol{p}$ in the second integral then leads to the standard form of the mode expansion of a one-component real Klein-Gordon field

$$\hat{\phi}(\boldsymbol{x},t) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}}\Big(\hat{a}_{\boldsymbol{p}}\, e^{-ip\cdot x} + \hat{a}^\dagger_{\boldsymbol{p}}\, e^{ip\cdot x}\Big), \quad \text{where} \quad p_0 = \omega_{\boldsymbol{p}}. \tag{7.14}$$

This is a general solution of the Klein-Gordon equation (7.4), which can be also derived by solving the Klein-Gordon equation directly (see Exercise 28). The two terms in the bracket are Hermitian conjugated so this expression for $\hat{\phi}(x)$ is clearly Hermitian.

The occurrence of two terms in the mode expansion (7.14) signals a qualitative difference between relativistic field theories and the non-relativistic Schrödinger field studied in Section 6.2, whose mode expansion is $\hat{\psi}(\boldsymbol{x},t) = \sum_k \hat{a}_k e^{-\frac{i}{\hbar}t\,\epsilon_k}u_k(\boldsymbol{x})$. This difference can be traced back to the time derivative term in the Lagrangian, which in the case of Klein-Gordon field, Eq. (7.1), reads $(\partial_t\phi)^2$, yielding a second-order-in-time (Klein-Gordon) equation of motion, whereas in the non-relativistic case, Eq. (6.24), the term $\psi^*\partial_t\psi$ yields an equation of motion that is of first-order in time derivative (the Schrödinger equation).

The operator-valued amplitudes $\hat{a}^\dagger_{\boldsymbol{p}}$ and $\hat{a}_{\boldsymbol{p}}$ are (time-independent) creation and annihilation operators of the respective modes $f_{\boldsymbol{p}}(\boldsymbol{x})$ whose commutation relations read (in analogy with the discrete case, Eq. (6.13))

$$[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p}'}] = \delta(\boldsymbol{p}-\boldsymbol{p}') \quad \text{and} \quad [\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{p}'}] = [\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p}'}] = 0. \tag{7.15}$$

The total Hamiltonian operator is given by an integral over all normal modes (cf. Eqs. (6.5) and (6.14)),

$$\hat{H} = \int d^3p \, \frac{\omega_{\boldsymbol{p}}}{2} \left(\hat{a}^\dagger_{\boldsymbol{p}} \hat{a}_{\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}} \hat{a}^\dagger_{\boldsymbol{p}} \right). \tag{7.16}$$

This result is confirmed in Exercise 28 by explicit substitution of the mode expansion of $\hat{\phi}(\boldsymbol{x}, t)$ into (quantized) formula (7.2). A use of the commutation relation $\hat{a}_{\boldsymbol{p}} \hat{a}^\dagger_{\boldsymbol{p}'} = \hat{a}^\dagger_{\boldsymbol{p}'} \hat{a}_{\boldsymbol{p}} + \delta(\boldsymbol{p} - \boldsymbol{p}')$ to formally rewrite

$$\hat{H} = \int d^3p \, \omega_{\boldsymbol{p}} \, \hat{a}^\dagger_{\boldsymbol{p}} \hat{a}_{\boldsymbol{p}} + \delta(\boldsymbol{0}) \int d^3p \, \frac{\omega_{\boldsymbol{p}}}{2} \tag{7.17}$$

reveals an infinite vacuum energy $\langle 0 | \hat{H} | 0 \rangle = \delta(\boldsymbol{0}) \int d^3p \, \frac{\omega_{\boldsymbol{p}}}{2}$. To avoid this contribution we define the *normal-ordered* Hamiltonian

$$:\hat{H}: = \hat{H} - \langle 0 | \hat{H} | 0 \rangle = \int d^3p \, \omega_{\boldsymbol{p}} \, \hat{a}^\dagger_{\boldsymbol{p}} \hat{a}_{\boldsymbol{p}}. \tag{7.18}$$

In general, the normal ordering $:\ldots:$ is a linear operation that pushes all creation operators to the left of annihilation operators, thus producing an operator with vanishing vacuum expectation value $\langle 0| :\ldots: |0\rangle$. However, there is a subtlety revealed by the following simple calculation for one degree of freedom ($[\hat{a}, \hat{a}^\dagger] = 1$):

$$:\hat{a}\hat{a}^\dagger: = \hat{a}^\dagger \hat{a} \neq :1 + \hat{a}^\dagger \hat{a}: = 1 + \hat{a}^\dagger \hat{a}, \tag{7.19}$$

i.e., one may not use commutation relations *under* normal ordering. Rather, all (bosonic) operators should be treated as commuting — they can be written in any order, since, after all, they are put to a specific order by the normal ordering.

Likewise, we normally order all components of the total four-momentum operator:

$$\hat{P}_\nu = \int d^3x \, :\hat{T}^0{}_\nu:, \quad \text{where} \quad T^\mu{}_\nu = \partial^\mu \phi \, \partial_\nu \phi - \delta^\mu_\nu \frac{1}{2} (\partial_\rho \phi \, \partial^\rho \phi - m^2 \phi^2) \tag{7.20}$$

is the Klein-Gordon energy-momentum tensor (5.65). For the zeroth component we have $\hat{P}_0 = :\hat{H}:$ given by Eq. (7.18), and for the spatial part we find (in Exercise 28)

$$\hat{P}_i = \int d^3x \, :\hat{\pi} \partial_i \hat{\phi}: = \int d^3p \, p_i \, \hat{a}^\dagger_{\boldsymbol{p}} \hat{a}_{\boldsymbol{p}}, \tag{7.21}$$

so in total we have

$$\hat{P}_\mu = \int d^3p \, p_\mu \, \hat{a}^\dagger_{\boldsymbol{p}} \hat{a}_{\boldsymbol{p}}, \quad \text{where} \quad p_0 = \omega_{\boldsymbol{p}}. \tag{7.22}$$

The elimination of infinite vacuum energy by means of normal ordering deserves a few comments. The divergent factor $\delta(\boldsymbol{0})$ can be related via

$$\delta(\boldsymbol{p}) = \int \frac{d^3x}{(2\pi)^3} \, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \quad \to \quad \delta(\boldsymbol{0}) = \int \frac{d^3x}{(2\pi)^3} = \frac{V}{(2\pi)^3} \tag{7.23}$$

to the volume V of the $\mathbb{R}^3$ space. One could regularize this divergence by using a box of edge length L instead of $\mathbb{R}^3$, and imposing periodic boundary conditions on the fields. (Periodic boundary conditions are easier to discuss than, for example, Dirichlet boundary conditions. In the end we are interested in the limit $L \to \infty$, when the details on the boundary should not matter.) The periodicity constrains the normal modes $f_{\boldsymbol{p}}(\boldsymbol{x}) \propto e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$ by the conditions $e^{ip^i L} = 1$ ($\forall i$), resulting in a discrete set of modes

$$f_{\boldsymbol{k}}(\boldsymbol{x}) = \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{L^{3/2}}, \quad \text{with} \quad \omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{p}^2 + m^2}, \quad \text{where} \quad \boldsymbol{p} = \frac{2\pi}{L}\boldsymbol{k} \qquad (\boldsymbol{k} \in \mathbb{Z}^3). \tag{7.24}$$

These are normalized to the (three-dimensional) Kronecker $\delta_{\boldsymbol{k},\boldsymbol{k}'} = \delta_{k_1,k_1'}\delta_{k_2,k_2'}\delta_{k_3,k_3'}$ instead of the Dirac $\delta(\boldsymbol{p}-\boldsymbol{p}')$, and so are the corresponding creation and annihilation operators:

$$\int d^3x\, f^*_{\boldsymbol{k}}(\boldsymbol{x}) f_{\boldsymbol{k}'}(\boldsymbol{x}) = \delta_{\boldsymbol{k},\boldsymbol{k}'} \quad , \quad [\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}. \tag{7.25}$$

In the large-L limit, identifying $\hat{a}_{\boldsymbol{p}} = L^{3/2}\hat{a}_{\boldsymbol{k}}$, the discrete Hamiltonian (formed by a sum over the discrete modes labelled by $\boldsymbol{k}$) turns into the integral over $\boldsymbol{p} \in \mathbb{R}^3$ in Eq. (7.16):

$$\hat{H} = \sum_{\boldsymbol{k}\in\mathbb{Z}^3} \frac{\omega_{\boldsymbol{k}}}{2}\left(\hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_{\boldsymbol{k}} + \hat{a}_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}\right) = \sum_{\boldsymbol{k}\in\mathbb{Z}^3} \frac{\omega_{\boldsymbol{p}}}{2L^3}\left(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}}\right)\Big|_{\boldsymbol{p}=\frac{2\pi}{L}\boldsymbol{k}} \xrightarrow{L\to\infty} \int d^3p\, \frac{\omega_{\boldsymbol{p}}}{2}\left(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}}\right). \tag{7.26}$$

By default we will formulate quantum field theory in infinite volume, keeping in mind that the finite-L 'box regularization' can be adopted if required.

In an infinite volume it is more sensible to talk about vacuum energy density

$$\frac{\langle 0|\, \hat{H}\, |0\rangle}{V} = \frac{\delta(\mathbf{0})}{(2\pi)^3\delta(\mathbf{0})}\int d^3p\, \frac{\omega_{\boldsymbol{p}}}{2} = \frac{1}{2(2\pi)^3}\int d^3p\, \sqrt{\boldsymbol{p}^2 + m^2}, \tag{7.27}$$

which, however, is still infinite due to the infinite range of integration. (In a finite box we would encounter an equally infinite sum over $\boldsymbol{k} \in \mathbb{Z}^3$.) This divergence could be tamed by introducing an upper limit on the value of spatial momentum $|\boldsymbol{p}|$ (an 'ultraviolet cut-off', usually denoted by Λ), which is equivalent to a minimum wavelength λ_{min} of the normal modes (due to the relation $|\boldsymbol{p}| = \frac{2\pi}{\lambda}$). A nonzero minimum wavelength would stem from a certain granularity (discreteness) of space. For example, if the space were a rectangular lattice with distance a between neighbouring points, $\lambda_{min} = 2a$ (see [9, Ch. 2.1.1]), and the integration would only involve momenta with $|\boldsymbol{p}| \leq \Lambda = \frac{\pi}{a}$. Here a is often considered to be of the order of *Planck length*

$$\ell_{Planck} = \sqrt{\frac{\hbar G}{c^3}} \doteq 1.6 \times 10^{-35}\,\mathrm{m} \doteq 10^{-20} \times (\text{proton diameter}). \tag{7.28}$$

A common practice in quantum field theory is to regard the space as a continuum and neglect the divergent momentum integral (7.27) by adopting normal ordering. (Later on we shall see that in interacting quantum field theory many other divergent momentum integrals appear, and these are dealt with by the method of *renormalization*.)

The constant vacuum energy (density) can be neglected if we are interested only in energy differences between various states, which will be the case in this course. However, in general relativity every energy, in principle, contributes to the Einstein equations and hence influences the spacetime dynamics. The quantum-field-theoretic vacuum contributes to Einstein's cosmological constant by an amount that is, theoretically, some 120 orders of magnitude larger than the value determined experimentally by observing the accelerated expansion of the universe. This discrepancy between theory and experiment is called the 'cosmological constant problem' — the "worst theoretical prediction in the history of science".

A more down-to-Earth aspect of the vacuum energy is that it refers to certain choice of boundary conditions, for example, the size of the box that the field occupies. If the boundary varies, the vacuum energy varies as well, and the system's tendency to minimize it results in a force exerted on the boundary. Although tiny, for quantized electromagnetic field constrained between two parallel conducting plates (so that the electric field components parallel to the plates vanish) this so-called *Casimir effect* has been measured in laboratory. In Exercise 30 we discuss a simplified version of this setup: one-component massless scalar field in one spatial dimension.

7.2 Multiplet of scalar fields

We will study a multicomponent Klein-Gordon field $\Phi = (\phi_r)$ described by the Lagrangian

$$\mathscr{L}(\phi_r, \partial_\mu \phi_r) = \sum_r \left(\frac{1}{2}(\partial_\mu \phi_r)(\partial^\mu \phi_r) - \frac{1}{2} m^2 \phi_r^2 \right) = \frac{1}{2}(\partial_\mu \Phi^T)(\partial^\mu \Phi) - \frac{1}{2} m^2 \Phi^T \Phi, \tag{7.29}$$

which is a sum of single-component Lagrangians (7.1). The individual (real) field components ϕ_r are independent of one another (there are no terms containing products of different components). Since all field components have the same mass parameter m this theory is invariant under internal rotations (generated by Hermitian purely imaginary matrices T_a)

$$\Phi'(x) = \exp\left(i\lambda_a \mathsf{T}_a\right) \Phi(x). \tag{7.30}$$

Canonical quantization proceeds similarly as in the one-component case. We find the canonical momenta, and postulate equal-time commutation relations:

$$\hat{\pi}_r = \partial_t \hat{\phi}_r \quad , \quad [\hat{\phi}_r(\boldsymbol{x}, t), \hat{\pi}_s(\boldsymbol{y}, t)] = i\, \delta_{rs}\, \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{7.31}$$

(The commutators involving other combinations of canonical fields vanish.) A general solution of the equations of motion $(\partial^\mu \partial_\mu + m^2)\hat{\phi}_r(x) = 0$ $(\forall r)$ has the form of a mode expansion

$$\hat{\phi}_r(\boldsymbol{x}, t) = \int \frac{d^3 p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}} \left(\hat{a}_{r,\boldsymbol{p}}\, e^{-ip\cdot x} + \hat{a}^\dagger_{r,\boldsymbol{p}}\, e^{ip\cdot x} \right) \quad , \quad p_0 = \omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}, \tag{7.32}$$

where the creation and annihilation operators (now labelled by the field component index as well as the spatial momentum) satisfy the commutation relations

$$[\hat{a}_{r,\boldsymbol{p}}, \hat{a}^\dagger_{s,\boldsymbol{p}'}] = \delta_{rs}\, \delta(\boldsymbol{p} - \boldsymbol{p}') \quad \text{and} \quad [\hat{a}_{r,\boldsymbol{p}}, \hat{a}_{s,\boldsymbol{p}'}] = [\hat{a}^\dagger_{r,\boldsymbol{p}}, \hat{a}^\dagger_{s,\boldsymbol{p}'}] = 0. \tag{7.33}$$

A new feature not present in the one-component case is the appearance of conserved currents and charges that we derived in Section 5.4.3 in classical context. In quantum theory they are the operators

$$\begin{aligned} \hat{j}^\mu_a &= -i \frac{\partial \hat{\mathscr{L}}}{\partial(\partial_\mu \phi_r)} (\mathsf{T}_a)_{rs} \hat{\phi}_s = -i(\partial^\mu \hat{\phi}_r)(\mathsf{T}_a)_{rs} \hat{\phi}_s, \\ \hat{Q}_a &= \int d^3x \ {:}\hat{j}^0_a{:} = -i \int d^3x \ {:}\hat{\pi}_r (\mathsf{T}_a)_{rs} \hat{\phi}_s{:}, \end{aligned} \tag{7.34}$$

where in the definition of the conserved charges we again include normal ordering, so that $\hat{Q}_a |0\rangle = 0$. Plugging in the mode expansion (7.32) one can show (see Exercise 31) that

$$\hat{Q}_a = \int d^3 p\, \hat{a}^\dagger_{r,\boldsymbol{p}} (\mathsf{T}_a)_{rs} \hat{a}_{s,\boldsymbol{p}}. \tag{7.35}$$

It is then interesting to note that the charge operators $\hat{Q}_a$ obey the same commutation relations as the rotation generators T_a (see Exercise 32):

$$[\mathsf{T}_a, \mathsf{T}_b] = f_{abc} \mathsf{T}_c \quad \Rightarrow \quad [\hat{Q}_a, \hat{Q}_b] = f_{abc} \hat{Q}_c. \tag{7.36}$$

7.2.1 Complex scalar field

Let us consider a two-component field Φ. Out of the two real components we can define a complex field φ (and its conjugate φ^*) as in Eq. (5.77):

$$\begin{pmatrix}\varphi\\ \varphi^*\end{pmatrix} = \mathsf{U}\Phi, \quad \text{where} \quad \Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \quad \text{and} \quad \mathsf{U} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix} \quad \text{is unitary.} \tag{7.37}$$

The Lagrangian (7.29) then takes the form

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\Phi^\dagger)(\partial^\mu\Phi) - \frac{m^2}{2}\Phi^\dagger\Phi = \frac{1}{2}\partial_\mu(\mathsf{U}\Phi)^\dagger\partial^\mu(\mathsf{U}\Phi) - \frac{m^2}{2}(\mathsf{U}\Phi)^\dagger\mathsf{U}\Phi = (\partial_\mu\varphi^*)(\partial^\mu\varphi) - m^2\varphi^*\varphi. \tag{7.38}$$

Correspondingly, we define the creation and annihilation operators (suppressing the label $\boldsymbol{p}$ for a moment)

$$\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix} = \mathsf{U}\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\hat{a}_1 + i\hat{a}_2\\ \hat{a}_1 - i\hat{a}_2\end{pmatrix} \quad \to \quad \begin{pmatrix}\hat{a}^\dagger\\ \hat{b}^\dagger\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\hat{a}_1^\dagger - i\hat{a}_2^\dagger\\ \hat{a}_1^\dagger + i\hat{a}_2^\dagger\end{pmatrix}, \tag{7.39}$$

and observe that the mode expansion of the complex field reads

$$\begin{aligned}\hat{\varphi}(\boldsymbol{x},t) &= \frac{1}{\sqrt{2}}\Big(\hat{\phi}_1(\boldsymbol{x},t) + i\hat{\phi}_2(\boldsymbol{x},t)\Big) = \int\frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}}\Big(\hat{a}_{\boldsymbol{p}}\, e^{-ip\cdot x} + \hat{b}^\dagger_{\boldsymbol{p}}\, e^{ip\cdot x}\Big),\\ \hat{\varphi}^\dagger(\boldsymbol{x},t) &= \int\frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}}\Big(\hat{b}_{\boldsymbol{p}}\, e^{-ip\cdot x} + \hat{a}^\dagger_{\boldsymbol{p}}\, e^{ip\cdot x}\Big).\end{aligned} \tag{7.40}$$

The creation and annihilation operators satisfy the commutation relations

$$[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p}'}] = [\hat{b}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p}'}] = \delta(\boldsymbol{p} - \boldsymbol{p}') \qquad \text{and} \qquad \text{all other commutators zero,} \tag{7.41}$$

which follows by a straightforward calculation. For example,

$$[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p}'}] = \frac{1}{2}[\hat{a}_{1,\boldsymbol{p}} + i\hat{a}_{2,\boldsymbol{p}}, \hat{a}^\dagger_{1,\boldsymbol{p}'} - i\hat{a}^\dagger_{2,\boldsymbol{p}'}] = \frac{1}{2}\Big([\hat{a}_{1,\boldsymbol{p}}, \hat{a}^\dagger_{1,\boldsymbol{p}'}] + [\hat{a}_{2,\boldsymbol{p}}, \hat{a}^\dagger_{2,\boldsymbol{p}'}]\Big) = \delta(\boldsymbol{p} - \boldsymbol{p}'). \tag{7.42}$$

The Lagrangian (7.38) is clearly invariant under the internal transformation $\varphi' = e^{i\lambda}\varphi$, $\varphi^{*\prime} = e^{-i\lambda}\varphi^*$. This is a $U(1)$ (complex) equivalent of an $SO(2)$ (two-component real) internal rotation by angle λ generated by the matrix

$$\mathsf{T} = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}, \quad \text{satisfying} \quad \mathsf{U}\mathsf{T}\mathsf{U}^\dagger = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{7.43}$$

(see Section 5.4.3). The Noether current corresponding to this internal symmetry is

$$j^\mu = -i\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi)}\varphi + i\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi^*)}\varphi^* = -i\varphi\,\partial^\mu\varphi^* + i\varphi^*\partial^\mu\varphi. \tag{7.44}$$

The total Noether charge operator $\hat{Q}$ expressed in terms of creation and annihilation operators can be read-off from Formula (7.35):

$$\hat{Q} = \int d^3p\,\begin{pmatrix}\hat{a}^\dagger_{1,\boldsymbol{p}} & \hat{a}^\dagger_{2,\boldsymbol{p}}\end{pmatrix}\mathsf{T}\begin{pmatrix}\hat{a}_{1,\boldsymbol{p}}\\ \hat{a}_{2,\boldsymbol{p}}\end{pmatrix} = \int d^3p\,\begin{pmatrix}\hat{a}^\dagger_{\boldsymbol{p}} & \hat{b}^\dagger_{\boldsymbol{p}}\end{pmatrix}\mathsf{U}\mathsf{T}\mathsf{U}^\dagger\begin{pmatrix}\hat{a}_{\boldsymbol{p}}\\ \hat{b}_{\boldsymbol{p}}\end{pmatrix} = \int d^3p\,\left(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} - \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}}\right). \tag{7.45}$$

Since the complex Klein-Gordon field possesses a conserved Noether charge, it is also referred to as 'charged Klein-Gordon field', while the real Klein-Gordon field is referred to as 'neutral Klein-Gordon field'.

Finally, let us note that the total four-momentum operator of the multicomponent field (ϕ_r) (with Lagrangian (7.29)) is given simply by a sum of the four-momentum operators (7.22) for each component (recall the general formula for the energy-momentum tensor (5.60)). In the present case of real two-component (or complex one-component) field this gives

$$\hat{P}_\mu = \int d^3p\, p_\mu \left(\hat{a}^\dagger_{1,\boldsymbol{p}}\hat{a}_{1,\boldsymbol{p}} + \hat{a}^\dagger_{2,\boldsymbol{p}}\hat{a}_{2,\boldsymbol{p}}\right) = \int d^3p\, p_\mu \left(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}}\right), \quad \text{where} \quad p_0 = \omega_{\boldsymbol{p}}. \tag{7.46}$$

Note that $[\hat{Q}, \hat{P}_\mu] = 0$, since all number operators mutually commute.

7.3 States and particle interpretation

The Klein-Gordon field is a continuous system of coupled oscillators whose normal modes are labelled by $\boldsymbol{p} \in \mathbb{R}^3$. Each mode behaves as a one-dimensional harmonic oscillator with frequency $\omega_{\boldsymbol{p}}$ whose nth energy level is reached by n-fold application of the creation operator $\hat{a}^\dagger_{\boldsymbol{p}}$ on the vacuum state $|0\rangle$ (recall Section 6.1, in particular, Eq. (6.8)). This creates a collective excitation of the field, whose spatial variation is given by the shape of the mode $f_{\boldsymbol{p}}(\boldsymbol{x}) \propto e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$. (For a multicomponent field, each component can be excited separately by its corresponding creation operator $\hat{a}^\dagger_{r,\boldsymbol{p}}$.)

The system's Hilbert space is spanned by the (collective) states that arise by applying all possible (finite) strings of creation operators on the vacuum, thus replicating the construction of the bosonic Fock space for non-relativistic Schrödinger field in Section 6.2. This suggests that the excited states of the collection of oscillators (i.e., of the quantum Klein-Gordon field) be interpreted as relativistic multi-particle states — adding a quantum of energy $\omega_{\boldsymbol{p}}$ in the mode $\boldsymbol{p}$ of the field component r is understood as adding a particle of 'type' r with energy $\omega_{\boldsymbol{p}}$ and spatial momentum $\boldsymbol{p}$.

The claim that the mode label $\boldsymbol{p}$ is to be interpreted as spatial momentum is supported by the following analysis of total four-momentum operator $\hat{P}_\mu$. (For definiteness we will consider the complex Klein-Gordon field $\hat{\varphi}$.) Expression (7.46) and the commutation relations (7.41) yield

$$[\hat{P}_\mu, \hat{a}^\dagger_{\boldsymbol{p}}] = \int d^3p'\, p'_\mu \left[\hat{a}^\dagger_{\boldsymbol{p}'}\hat{a}_{\boldsymbol{p}'} + \hat{b}^\dagger_{\boldsymbol{p}'}\hat{b}_{\boldsymbol{p}'}, \hat{a}^\dagger_{\boldsymbol{p}}\right] = \int d^3p'\, p'_\mu \hat{a}^\dagger_{\boldsymbol{p}'}\delta(\boldsymbol{p}' - \boldsymbol{p}) = p_\mu \hat{a}^\dagger_{\boldsymbol{p}}, \tag{7.47}$$

and similarly for $\hat{b}^\dagger_{\boldsymbol{p}}$. Hermitian conjugation (keeping in mind that $\hat{P}^\dagger_\mu = \hat{P}_\mu$) produces the corresponding relations for annihilation operators. Altogether,

$$[\hat{P}_\mu, \hat{a}^\dagger_{\boldsymbol{p}}] = p_\mu \hat{a}^\dagger_{\boldsymbol{p}} \quad , \quad [\hat{P}_\mu, \hat{a}_{\boldsymbol{p}}] = -p_\mu \hat{a}_{\boldsymbol{p}} \qquad \text{and} \qquad [\hat{P}_\mu, \hat{b}^\dagger_{\boldsymbol{p}}] = p_\mu \hat{b}^\dagger_{\boldsymbol{p}} \quad , \quad [\hat{P}_\mu, \hat{b}_{\boldsymbol{p}}] = -p_\mu \hat{b}_{\boldsymbol{p}}, \tag{7.48}$$

where, just to remind, $p_0 = \omega_{\boldsymbol{p}}$. These relations, and the fact that $\hat{P}_\mu$ is normal-ordered (so that $\hat{P}_\mu |0\rangle = 0$), allow us to easily determine (using the commutator Leibniz rule) the total four-momentum of a state created from the vacuum by an arbitrary string of operators $\hat{a}^\dagger_{\boldsymbol{p}}$, $\hat{b}^\dagger_{\boldsymbol{p}}$:

$$\begin{aligned}\hat{P}^\mu \hat{a}^\dagger_{\boldsymbol{p}_1} \cdots \hat{a}^\dagger_{\boldsymbol{p}_{n_a}} \hat{b}^\dagger_{\boldsymbol{p}'_1} \cdots \hat{b}^\dagger_{\boldsymbol{p}'_{n_b}} |0\rangle &= \left[\hat{P}^\mu, \hat{a}^\dagger_{\boldsymbol{p}_1} \cdots \hat{a}^\dagger_{\boldsymbol{p}_{n_a}} \hat{b}^\dagger_{\boldsymbol{p}'_1} \cdots \hat{b}^\dagger_{\boldsymbol{p}'_{n_b}}\right] |0\rangle \\ &= (p^\mu_1 + \cdots + p^\mu_{n_a} + p'^\mu_1 + \cdots + p'^\mu_{n_b})\, \hat{a}^\dagger_{\boldsymbol{p}_1} \cdots \hat{a}^\dagger_{\boldsymbol{p}_{n_a}} \hat{b}^\dagger_{\boldsymbol{p}'_1} \cdots \hat{b}^\dagger_{\boldsymbol{p}'_{n_b}} |0\rangle\,.\end{aligned} \tag{7.49}$$

The spatial part of this relation ($\mu = i$) asserts that adding an excitation to mode $\boldsymbol{p}$ increases the total momentum by the amount $\boldsymbol{p}$, i.e., a particle with momentum $\boldsymbol{p}$ is added to the system.

When we combine the knowledge of relations (7.48) with the mode expansion (7.40), we obtain

$$[\hat{P}_\mu, \hat{\varphi}(x)] = \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}} \Big([\hat{P}_\mu, \hat{a}_{\boldsymbol{p}}]\, e^{-ip\cdot x} + [\hat{P}_\mu, \hat{b}^\dagger_{\boldsymbol{p}}]\, e^{ip\cdot x} \Big) = -i\partial_\mu \hat{\varphi}(x), \tag{7.50}$$

which for $\mu = 0$ recovers the Heisenberg equation of motion

$$\partial_t \hat{\varphi}(x) = -i[\hat{\varphi}(x), \hat{H}]. \tag{7.51}$$

Contracting Eq. (7.50) with a spacetime vector a^μ, and using the Campbell identity (2.14), we relate the field operator $\hat{\varphi}$ at different spacetime points:

$$[ia^\mu \hat{P}_\mu, \hat{\varphi}(x)] = a^\mu \partial_\mu \hat{\varphi}(x) \quad \rightarrow \quad e^{ia^\mu \hat{P}_\mu} \hat{\varphi}(x) e^{-ia^\mu \hat{P}_\mu} = \sum_{n=0}^{\infty} \frac{(a^\mu \partial_\mu)^n}{n!}\, \hat{\varphi}(x) = \hat{\varphi}(x+a). \tag{7.52}$$

(Here we note that the infinite series is actually the Taylor expansion of $\hat{\varphi}$ around point x.)

We can make similar observations for the charge operator $\hat{Q}$ (given by (7.45)) in place of $\hat{P}_\mu$. First, we find the commutators

$$[\hat{Q}, \hat{a}^\dagger_{\boldsymbol{p}}] = \hat{a}^\dagger_{\boldsymbol{p}} \quad , \quad [\hat{Q}, \hat{a}_{\boldsymbol{p}}] = -\hat{a}_{\boldsymbol{p}} \qquad \text{and} \qquad [\hat{Q}, \hat{b}^\dagger_{\boldsymbol{p}}] = -\hat{b}^\dagger_{\boldsymbol{p}} \quad , \quad [\hat{Q}, \hat{b}_{\boldsymbol{p}}] = \hat{b}_{\boldsymbol{p}}. \tag{7.53}$$

From the mode expansion (7.40) and the Campbell identity (2.14) we then obtain the relation

$$[\hat{Q}, \hat{\varphi}(x)] = -\hat{\varphi}(x) \quad \rightarrow \quad e^{i\lambda\hat{Q}} \hat{\varphi}(x) e^{-i\lambda\hat{Q}} = \sum_{n=0}^{\infty} \frac{(-i\lambda)^n}{n!}\, \hat{\varphi}(x) = e^{-i\lambda} \hat{\varphi}(x). \tag{7.54}$$

The total charge of a state created from the vacuum by a string of operators $\hat{a}^\dagger_{\boldsymbol{p}}$, $\hat{b}^\dagger_{\boldsymbol{p}}$ reads

$$\hat{Q}\, \hat{a}^\dagger_{\boldsymbol{p}_1} \cdots \hat{a}^\dagger_{\boldsymbol{p}_{n_a}} \hat{b}^\dagger_{\boldsymbol{p}'_1} \cdots \hat{b}^\dagger_{\boldsymbol{p}'_{n_b}} \left|0\right\rangle = (n_a - n_b)\, \hat{a}^\dagger_{\boldsymbol{p}_1} \cdots \hat{a}^\dagger_{\boldsymbol{p}_{n_a}} \hat{b}^\dagger_{\boldsymbol{p}'_1} \cdots \hat{b}^\dagger_{\boldsymbol{p}'_{n_b}} \left|0\right\rangle, \tag{7.55}$$

where we have used the fact that $\hat{Q}\left|0\right\rangle = 0$, and the Leibniz rule. This result can be easily deduced already from Eq. (7.45), since $\hat{a}^\dagger_{\boldsymbol{p}} \hat{a}_{\boldsymbol{p}}$ is the number operator for particles of type a with momentum $\boldsymbol{p}$, and so $\hat{n}_a = \int d^3p\, \hat{a}^\dagger_{\boldsymbol{p}} \hat{a}_{\boldsymbol{p}}$ is the number operator for the type-a particles regardless of their momenta. Similarly, for type-b particles the number operator is $\hat{n}_b = \int d^3p\, \hat{b}^\dagger_{\boldsymbol{p}} \hat{b}_{\boldsymbol{p}}$, and hence $\hat{Q} = \hat{n}_a - \hat{n}_b$.

The operators $\hat{a}^\dagger_{\boldsymbol{p}}$ create a particle with charge $+1$, and four-momentum $(\omega_{\boldsymbol{p}}, \boldsymbol{p})$, while the operator $\hat{b}^\dagger_{\boldsymbol{p}}$ creates particles with charge -1, and the same four-momentum $(\omega_{\boldsymbol{p}}, \boldsymbol{p})$. The b particles therefore are called the *antiparticles* of the a particles. Particles described by a real-valued Klein-Gordon field do not have antiparticles (one may as well say that they "are their own antiparticles"). In particle physics, a charged scalar field describes, for example, π-mesons (pions) π^+ and π^-, while a neutral scalar field describes the pion π^0. These are mediators of the nuclear force (the residual strong force) that holds protons and neutrons together in a nucleus. In this example the Noether charge is the usual electric charge. Nevertheless, even electrically neutral particles can have antiparticles because other types of charges can arise from internal symmetries. For example, the electrically neutral K meson (kaon) K^0 has an antiparticle $\bar{K}^0$ with opposite 'strangeness'.

In this section we have given a particle interpretation to the quantized Klein-Gordon field, which we originally introduced merely as a continuous system of coupled oscillators. This approach is quite the opposite of what was done in Section 6.2, i.e., the 'second quantization', where we started with a system of quantum-mechanical particles and developed the formalism of non-relativistic quantum field theory to describe them more efficiently.

In general, it is really the concept of a quantum field that is more fundamental. (Quantum field theory is a quantum theory of *fields*, not particles.) For free field theories (i.e., quadratic Lagrangians) the normal mode decomposition allows to set up a particle interpretation of states, however, for interacting field theories (containing higher than quadratic terms in the Lagrangian) normal modes can be used only as an approximation and particle interpretation becomes problematic. Still, in most cases we will encounter, the interaction terms can be treated as perturbations added to the free Lagrangian, and the particle interpretation is justified (and indeed useful) at least on the perturbative level.

Another issue with the particle interpretation is that it depends on an observer. What an inertial observer in Minkowski spacetime calls the vacuum is in fact a state that an observer moving with acceleration a describes as a thermal bath at temperature T, where the mean number of particles with energy E is given by the Bose-Einstein distribution

$$n(E) = \frac{1}{\exp\left(\frac{E}{k_B T}\right) - 1}, \quad \text{where} \quad \frac{T}{a} = \frac{\hbar}{2\pi c k_B} \doteq 4 \times 10^{-21} K. \tag{7.56}$$

This so-called *Unruh effect* (to be discussed in detail in Chapter 13.2) is rather small and so far has not been experimentally detected; 1 Kelvin corresponds to an acceleration of the order $10^{20}\,\mathrm{ms}^{-2}$.

7.3.1 Relativistic normalisation and one-particle states

Before closing the chapter on Klein-Gordon field let us make a few observations about relativistic particle states and their normalization.

Consider a Lorentz-invariant function $f(p)$, and write an integral

$$\int d^4p\, \theta(p_0)\delta(p^2 - m^2) f(p) = \int \frac{d^3p}{2\omega_{\boldsymbol{p}}} f(\omega_{\boldsymbol{p}}, \boldsymbol{p}), \tag{7.57}$$

where θ is the Heaviside step function, and we employed formula (7.72). Since the left-hand side is invariant under (proper orthochronous) Lorentz transformations, we identify a Lorentz-invariant three-dimensional integration measure $\frac{d^3p}{2\omega_{\boldsymbol{p}}}$. Moreover, writing

$$1 = \int d^3p\, \delta(\boldsymbol{p} - \boldsymbol{q}) = \int \frac{d^3p}{2\omega_{\boldsymbol{p}}}\, 2\omega_{\boldsymbol{p}} \delta(\boldsymbol{p} - \boldsymbol{q}) \tag{7.58}$$

then reveals a Lorentz-invariant modification of the three-dimensional delta function, $2\omega_{\boldsymbol{p}}\delta(\boldsymbol{p}-\boldsymbol{q})$.

The one-particle relativistic wave-function corresponding to a state $\hat{a}^\dagger_{\boldsymbol{p}} |0\rangle$ is constructed similarly as in the non-relativistic case, Eq. (6.39),

$$\langle 0| \hat{\phi}(x) \hat{a}^\dagger_{\boldsymbol{p}} |0\rangle = \int \frac{d^3p'}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}'}}} \langle 0| \hat{a}_{\boldsymbol{p}'} \hat{a}^\dagger_{\boldsymbol{p}} |0\rangle\, e^{-ip'\cdot x} = \frac{e^{-ip\cdot x}}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}}. \tag{7.59}$$

(Here $\hat{\phi}$ is a real field for simplicity, but the discussion extends to multicomponent or complex Klein-Gordon fields in a straightforward manner.) It is convenient to modify the normalization and define the one-particle states

$$|\boldsymbol{p}\rangle \equiv \sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}\, \hat{a}^\dagger_{\boldsymbol{p}} |0\rangle\,, \quad \text{such that} \quad \langle 0| \hat{\phi}(x) |\boldsymbol{p}\rangle = e^{-ip\cdot x}. \tag{7.60}$$

These states are normalised in a relativistically invariant way:

$$\langle \boldsymbol{p}|\boldsymbol{q}\rangle = \sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}\sqrt{(2\pi)^3 2\omega_{\boldsymbol{q}}}\,\langle 0|\,\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{q}}\,|0\rangle = (2\pi)^3 2\omega_{\boldsymbol{p}}\delta(\boldsymbol{p}-\boldsymbol{q}). \tag{7.61}$$

For multi-particle states $|\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots\rangle$ we include for each particle with momentum $\boldsymbol{p}$ a factor $\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}$, and should there be several, say n, particles with the same momentum we also multiply by $\frac{1}{\sqrt{n!}}$ (see Eq. (6.40)).

Finally, observe that if the field operator $\hat{\phi}(x)$ satisfies the Klein-Gordon field equation of motion (7.4), then the one-particle wave-function $\langle 0|\,\hat{\phi}(x)\,|\alpha\rangle$, where

$$|\alpha\rangle = \int d^3p\,\alpha(\boldsymbol{p})\,|\boldsymbol{p}\rangle \tag{7.62}$$

is a wave-packet characterized by an arbitrary function $\alpha(\boldsymbol{p})$, automatically satisfies the Klein-Gordon wave equation (1.19):

$$(\partial^\mu\partial_\mu + m^2)\hat{\phi}(x) = 0 \quad \rightarrow \quad (\partial^\mu\partial_\mu + m^2)\,\langle 0|\,\hat{\phi}(x)\,|\alpha\rangle = 0. \tag{7.63}$$

The field dynamics, which was determined by our choice of Lagrangian (7.1), therefore correctly reproduces the one-particle dynamics established within relativistic quantum mechanics. Such correspondence is an important guiding principle for the choice of quantum field theory Lagrangians.

7.4 Exercises

Exercise 28. *Mode expansion from Klein-Gordon equation.* Write a general solution of the (operator-valued) Klein-Gordon equation

$$(\partial^\mu \partial_\mu + m^2)\hat{\phi}(x) = 0 \tag{7.64}$$

as a superposition of plane waves, and recover the mode expansion (7.14).

Solution:

A plane wave that satisfies the Klein-Gordon equation is of the form

$$\hat{A}(p)\, e^{-ip\cdot x}, \quad \text{where} \quad p^\mu p_\mu = m^2. \tag{7.65}$$

A general superposition of these waves (that respects the dispersion relation) reads

$$\hat{\phi}(x) = \int d^4p\, \delta(p^\mu p_\mu - m^2)\hat{A}(p)\, e^{-ip\cdot x}. \tag{7.66}$$

Now recall that for a function $f(t)$ composed with δ-function the following identity holds:

$$\delta(f(t)) = \sum_i \frac{\delta(t - t_i)}{|f'(t_i)|}, \tag{7.67}$$

where the sum extends over all roots of f, $f(t_i) = 0$, and it is assumed that $f'(t_i) \neq 0$. To understand this formula, note that $\delta(f(t))$ vanishes unless $f(t) = 0$, and in the neighbourhood I_i of a zero point t_i we can make use of the Taylor expansion of f,

$$\delta(f(t)) = \delta(\underbrace{f(t_i)}_{=0} + f'(t_i)(t - t_i) + \ldots) = \frac{\delta(t - t_i)}{|f'(t_i)|} \qquad (t \in I_i). \tag{7.68}$$

Summing the right-hand side over i we obtain a function that equals $\delta(f(t))$ for all $t \in \mathbb{R}$, hence the identity (7.67). More rigorously, since f is monotonous on each (small enough) interval I_i, taking an arbitrary test function φ we can write

$$\int_{\mathbb{R}} dt\, \delta(f(t))\varphi(t) = \sum_i \int_{I_i} dt\, \delta(f(t))\varphi(t), \tag{7.69}$$

and perform the substitution $u = f(t)$ in each of the sub-integrals:

$$\int_{I_i} dt\, \delta(f(t))\varphi(t) = \int_{f(I_i)} du\, \frac{\delta(u)}{|f'(f^{-1}(u))|}\varphi(f^{-1}(u)) = \frac{\varphi(t_i)}{|f'(t_i)|} = \int_{\mathbb{R}} dt\, \frac{\delta(t - t_i)}{|f'(t_i)|}\varphi(t). \tag{7.70}$$

Since φ is arbitrary we obtain the equality (7.67).

In Eq. (7.66) we have (denoting $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$)

$$f(p_0) = p^\mu p_\mu - m^2 = p_0^2 - \omega_{\boldsymbol{p}}^2 = (p_0 - \omega_{\boldsymbol{p}})(p_0 + \omega_{\boldsymbol{p}}) \quad , \quad f'(p_0) = 2p_0, \tag{7.71}$$

and so

$$\delta(p^\mu p_\mu - m^2) = \frac{\delta(p_0 - \omega_{\boldsymbol{p}})}{2\omega_{\boldsymbol{p}}} + \frac{\delta(p_0 + \omega_{\boldsymbol{p}})}{2\omega_{\boldsymbol{p}}}. \tag{7.72}$$

Hence, we can carry out the p_0 integration and find

$$\begin{aligned}\hat{\phi}(x) &= \int \frac{d^3p}{2\omega_{\boldsymbol{p}}} \int dp_0 \left(\delta(p_0 - \omega_{\boldsymbol{p}}) + \delta(p_0 + \omega_{\boldsymbol{p}})\right) \hat{A}(p)\, e^{-ip\cdot x} \\ &= \int \frac{d^3p}{2\omega_{\boldsymbol{p}}} \left(\hat{A}(\omega_{\boldsymbol{p}}, \boldsymbol{p})\, e^{-i\omega_{\boldsymbol{p}} t + i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{A}(-\omega_{\boldsymbol{p}}, \boldsymbol{p})\, e^{i\omega_{\boldsymbol{p}} t + i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \\ &= \int \frac{d^3p}{2\omega_{\boldsymbol{p}}} \left(\hat{A}(\omega_{\boldsymbol{p}}, \boldsymbol{p})\, e^{-ip\cdot x} + \hat{A}(-\omega_{\boldsymbol{p}}, -\boldsymbol{p})\, e^{ip\cdot x}\right), \end{aligned} \tag{7.73}$$

where in the last expression $p_0 = \omega_{\boldsymbol{p}}$. For $\hat{\phi}(x)$ to be Hermitian we must have $\hat{A}(-\omega_{\boldsymbol{p}}, -\boldsymbol{p}) = \hat{A}^\dagger(\omega_{\boldsymbol{p}}, \boldsymbol{p})$. The identification

$$\hat{A}(\omega_{\boldsymbol{p}}, \boldsymbol{p}) = \sqrt{\frac{2\omega_{\boldsymbol{p}}}{(2\pi)^3}}\, \hat{a}_{\boldsymbol{p}} \tag{7.74}$$

then recovers the mode expansion of the Klein-Gordon field, Eq. (7.14).

Exercise 29. *Total four-momentum in terms of creation and annihilation operators.* Using the mode expansion of the Klein-Gordon field,

$$\hat{\phi}(\boldsymbol{x}, t) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}} \left(\hat{a}_{\boldsymbol{p}}\, e^{-ip\cdot x} + \hat{a}^\dagger_{\boldsymbol{p}}\, e^{ip\cdot x}\right) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}} \left(\hat{a}_{\boldsymbol{p}}\, e^{-i\omega_{\boldsymbol{p}} t + i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{\boldsymbol{p}}\, e^{i\omega_{\boldsymbol{p}} t - i\boldsymbol{p}\cdot\boldsymbol{x}}\right), \tag{7.75}$$

derive the formula (7.22) for the (normal-ordered) total four-momentum operator $\hat{P}_\mu$.

Solution:

Since $\hat{P}_\mu$ is time-independent (it is a conserved quantity), we may choose to evaluate it at time $t = 0$ for convenience. We will use the mode expansions

$$\begin{aligned}\hat{\phi}(\boldsymbol{x}, 0) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}} \left(\hat{a}_{\boldsymbol{p}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{\boldsymbol{p}}\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \\ \hat{\pi}(\boldsymbol{x}, 0) = \partial_t \hat{\phi}(\boldsymbol{x}, 0) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}} \left(-i\omega_{\boldsymbol{p}}\, \hat{a}_{\boldsymbol{p}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + i\omega_{\boldsymbol{p}}\, \hat{a}^\dagger_{\boldsymbol{p}}\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \\ \partial_i \hat{\phi}(\boldsymbol{x}, 0) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}} \left(ip^i\, \hat{a}_{\boldsymbol{p}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - ip^i\, \hat{a}^\dagger_{\boldsymbol{p}}\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \end{aligned} \tag{7.76}$$

to calculate the integrals

$$\begin{aligned} &1. \quad \int d^3x\, \hat{T}^0{}_i = \int d^3x\, \hat{\pi}\partial_i\hat{\phi}, \\ &2. \quad \int d^3x\, \hat{T}^0{}_0 = \int d^3x\, \frac{1}{2}\left(\hat{\pi}^2 + (\partial_i\hat{\phi})(\partial_i\hat{\phi}) + m^2\hat{\phi}^2\right), \end{aligned} \tag{7.77}$$

and then apply normal ordering. Before starting, let us recall the integral formula $\int d^3x\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} = (2\pi)^3\delta(\boldsymbol{p})$.

1.

$$\begin{aligned}\int d^3x\,\hat{\pi}\partial_i\hat{\phi} &= \int \frac{d^3x\,d^3p\,d^3p'\,\omega_{\boldsymbol{p}}\,p'^i}{(2\pi)^3\sqrt{2\omega_{\boldsymbol{p}}2\omega_{\boldsymbol{p}'}}}\Big(\hat{a}_{\boldsymbol{p}}\,e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - \hat{a}^\dagger_{\boldsymbol{p}}\,e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\Big)\Big(\hat{a}_{\boldsymbol{p}'}\,e^{i\boldsymbol{p}'\cdot\boldsymbol{x}} - \hat{a}^\dagger_{\boldsymbol{p}'}\,e^{-i\boldsymbol{p}'\cdot\boldsymbol{x}}\Big)\\
&= \int \frac{d^3p\,d^3p'\,\omega_{\boldsymbol{p}}\,p'^i}{\sqrt{2\omega_{\boldsymbol{p}}2\omega_{\boldsymbol{p}'}}}\Big(\hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}\delta(\boldsymbol{p}+\boldsymbol{p}') - \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}\delta(\boldsymbol{p}-\boldsymbol{p}')\\
&\qquad - \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}\delta(-\boldsymbol{p}+\boldsymbol{p}') + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}\delta(-\boldsymbol{p}-\boldsymbol{p}')\Big)\\
&= \int \frac{d^3p\,\omega_{\boldsymbol{p}}}{\sqrt{2\omega_{\boldsymbol{p}}2\omega_{\boldsymbol{p}}}}\Big(-p^i\hat{a}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}} - p^i\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} - p^i\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} - p^i\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}}\Big)\\
&= \int d^3p\,\frac{p_i}{2}\Big(\hat{a}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}}\Big) \qquad (7.78)\end{aligned}$$

Now, $\hat{a}_{\boldsymbol{p}}$ commutes with $\hat{a}_{-\boldsymbol{p}}$, and so, by substituting $\boldsymbol{p}\to-\boldsymbol{p}$,

$$\int d^3p\,p_i\,\hat{a}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}} = \int d^3p\,(-p_i)\hat{a}_{-\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} = -\int d^3p\,p_i\,\hat{a}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}} = 0, \qquad (7.79)$$

and similarly for the $\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}}$ term. Hence we arrive at

$$\int d^3x\,\hat{\pi}\partial_i\hat{\phi} = \int d^3p\,\frac{p_i}{2}\Big(\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}}\Big) \quad\rightarrow\quad \hat{P}_i = \int d^3x\;:\hat{\pi}\partial_i\hat{\phi}:\, = \int d^3p\,p_i\,\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} \quad (7.80)$$

2. By the same methods we can derive the following:

$$\begin{aligned}\int d^3x\,\hat{\pi}^2 &= \int \frac{d^3p}{2\omega_{\boldsymbol{p}}}\,\omega^2_{\boldsymbol{p}}\Big(-\hat{a}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} - \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}}\Big),\\
\int d^3x\,(\partial_i\hat{\phi})(\partial_i\hat{\phi}) &= \int \frac{d^3p}{2\omega_{\boldsymbol{p}}}\,\boldsymbol{p}^2\Big(\hat{a}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}}\Big),\\
\int d^3x\,m^2\hat{\phi}^2 &= \int \frac{d^3p}{2\omega_{\boldsymbol{p}}}\,m^2\Big(\hat{a}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}}\Big). \qquad (7.81)\end{aligned}$$

Summing up (including the factor $\frac{1}{2}$), and taking into account the relation $\omega^2_{\boldsymbol{p}} = \boldsymbol{p}^2 + m^2$, we obtain

$$\hat{H} = \int d^3p\,\frac{\omega_{\boldsymbol{p}}}{2}\Big(\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}}\Big) \quad\rightarrow\quad \hat{P}_0 = \,:\hat{H}:\, = \int d^3p\,\omega_{\boldsymbol{p}}\,\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}}. \qquad (7.82)$$

Exercise 30. *Casimir effect.* Consider a one-component massless Klein-Gordon field ϕ in one spatial dimension, which is constraint to vanish at points $x = 0, d, L$. What is the dependence of the vacuum energy on the position of the intermediate point d?

Solution:

The field modes between points 0 and d (region I) that solve the equation $(\partial_t^2 - \partial_x^2)\phi = 0$ and respect the boundary conditions $\phi(0) = \phi(d) = 0$ are given by

$$f_k(x) \propto \sin(p_k x), \quad \text{with} \quad \sin(p_k d) = 0 \quad\rightarrow\quad p_k = \frac{k\pi}{d} \quad (k = 1, 2, \ldots), \qquad (7.83)$$

and the respective energies (for the massless field) read $\omega_k = |p_k| = \frac{k\pi}{d}$. The vacuum energy between 0 and d is therefore

$$\tilde{E}_I(d) = \sum_{k=1}^{\infty} \frac{\omega_k}{2} = \frac{\pi}{2d} \sum_{k=1}^{\infty} k, \tag{7.84}$$

which is very infinite. This infinity can be regularized by introducing additional factors $e^{-a\omega_k}$ (which are relatively easy to handle analytically), so that frequencies $\omega_k \gg \frac{1}{a}$ effectively do not contribute to the vacuum energy. (In an actual experiment with electromagnetic field the plates do not have infinite conductivity, therefore they become transparent to electromagnetic waves of high-enough frequency.) Then we can calculate

$$\begin{aligned} E_I(d) &= \frac{\pi}{2d} \sum_{k=1}^{\infty} k \exp\Big(- \frac{a\pi}{d} k\Big) = -\frac{\pi}{2d} \frac{d}{\pi} \frac{\partial}{\partial a} \sum_{k=0}^{\infty} \exp\Big(- \frac{a\pi}{d} k\Big) = -\frac{1}{2} \frac{\partial}{\partial a} \frac{1}{1 - \exp\big(- \frac{a\pi}{d}\big)} \\ &= \frac{\pi}{2d} \frac{\exp\big(- \frac{a\pi}{d}\big)}{\big[1 - \exp\big(- \frac{a\pi}{d}\big)\big]^2} = \frac{\pi}{8d} \frac{1}{\sinh^2(\frac{a\pi}{2d})}. \end{aligned} \tag{7.85}$$

Since we eventually want to drop the regularisation, we look for an expansion as $a \to 0$:

$$\begin{aligned} E_I(d) &= \frac{d}{2\pi a^2} \left. \frac{z^2}{\sinh^2(z)} \right|_{z=\frac{a\pi}{2d}} = \frac{d}{2\pi a^2} \left. \frac{1}{\big(1 + \frac{z^2}{3!} + \mathcal{O}(z^4)\big)^2} \right|_{z=\frac{a\pi}{2d}} = \frac{d}{2\pi a^2} \left. \left(1 - \frac{z^2}{3} + \mathcal{O}(z^4)\right) \right|_{z=\frac{a\pi}{2d}} \\ &= \frac{d}{2\pi a^2} - \frac{\pi}{24d} + \mathcal{O}(a^2). \end{aligned} \tag{7.86}$$

By analogy, we have for the modes between d and L (region II) the regularised energy $E_{II}(d) = E_I(L - d)$, and so the total energy reads

$$E(d) = E_I(d) + E_{II}(d) = \frac{L}{2\pi a^2} - \frac{\pi}{24} \left(\frac{1}{d} + \frac{1}{L - d} \right) + \mathcal{O}(a^2) \tag{7.87}$$

The force $F(d)$ exerted on the middle point (or plate in three dimensions) is calculated as the negative gradient of the total energy:

$$F(d) = -\frac{\partial E}{\partial d} = -\frac{\pi}{24} \left(\frac{1}{d^2} - \frac{1}{(L - d)^2} \right) + \mathcal{O}(a^2). \tag{7.88}$$

Interestingly, we can now send $a \to 0$, and obtain a finite result even without the regularizing parameter. If the middle point is close to the origin ($L \gg d$) we obtain the *Casimir force*

$$F(d) \approx -\frac{\pi \hbar c}{24 d^2}, \tag{7.89}$$

where we reintroduced $\hbar$ and c. The Casimir force is attractive, and (being proportional to $\hbar$) very tiny for macroscopic setups.

Exercise 31. *Conserved charges in terms of creation and annihilation operators.* Using the mode expansion of a multicomponent Klein-Gordon field,

$$\hat{\phi}_r(\boldsymbol{x}, t) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}} \Big(\hat{a}_{r,\boldsymbol{p}}\, e^{-ip\cdot x} + \hat{a}^\dagger_{r,\boldsymbol{p}}\, e^{ip\cdot x} \Big), \tag{7.90}$$

derive the expression (7.35) for (normal-ordered) charge operators

$$\hat{Q}_a = -i \int d^3x \ :\hat{\pi}_r (\mathsf{T}_a)_{rs} \hat{\phi}_s: \ . \tag{7.91}$$

Solution:

Following the same route as in Exercise 29 we find

$$\begin{aligned}
\int d^3x\, \hat{\pi}_r \hat{\phi}_s &= \int \frac{d^3x\, d^3p\, d^3p'}{(2\pi)^3 \sqrt{2\omega_{\boldsymbol{p}} 2\omega_{\boldsymbol{p}'}}} \Big(- i\omega_{\boldsymbol{p}}\, \hat{a}_{r,\boldsymbol{p}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + i\omega_{\boldsymbol{p}}\, \hat{a}^\dagger_{r,\boldsymbol{p}}\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \Big) \Big(\hat{a}_{s,\boldsymbol{p}'}\, e^{i\boldsymbol{p}'\cdot\boldsymbol{x}} + \hat{a}^\dagger_{s,\boldsymbol{p}'}\, e^{-i\boldsymbol{p}'\cdot\boldsymbol{x}} \Big) \\
&= \int \frac{d^3p\, d^3p'\, i\omega_{\boldsymbol{p}}}{\sqrt{2\omega_{\boldsymbol{p}} 2\omega_{\boldsymbol{p}'}}} \Big(- \hat{a}_{r,\boldsymbol{p}} \hat{a}_{s,\boldsymbol{p}'} \delta(\boldsymbol{p}+\boldsymbol{p}') - \hat{a}_{r,\boldsymbol{p}} \hat{a}^\dagger_{s,\boldsymbol{p}'} \delta(\boldsymbol{p}-\boldsymbol{p}') \\
&\qquad + \hat{a}^\dagger_{r,\boldsymbol{p}} \hat{a}_{s,\boldsymbol{p}'} \delta(-\boldsymbol{p}+\boldsymbol{p}') + \hat{a}^\dagger_{r,\boldsymbol{p}} \hat{a}^\dagger_{s,\boldsymbol{p}'} \delta(-\boldsymbol{p}-\boldsymbol{p}') \Big) \\
&= \int \frac{d^3p\, i\omega_{\boldsymbol{p}}}{\sqrt{2\omega_{\boldsymbol{p}} 2\omega_{\boldsymbol{p}}}} \Big(- \hat{a}_{r,\boldsymbol{p}} \hat{a}_{s,-\boldsymbol{p}} - \hat{a}_{r,\boldsymbol{p}} \hat{a}^\dagger_{s,\boldsymbol{p}} + \hat{a}^\dagger_{r,\boldsymbol{p}} \hat{a}_{s,\boldsymbol{p}} + \hat{a}^\dagger_{r,\boldsymbol{p}} \hat{a}^\dagger_{s,-\boldsymbol{p}} \Big) \\
&= \int d^3p\, \frac{i}{2} \Big(- \hat{a}_{r,\boldsymbol{p}} \hat{a}_{s,-\boldsymbol{p}} - \hat{a}_{r,\boldsymbol{p}} \hat{a}^\dagger_{s,\boldsymbol{p}} + \hat{a}^\dagger_{r,\boldsymbol{p}} \hat{a}_{s,\boldsymbol{p}} + \hat{a}^\dagger_{r,\boldsymbol{p}} \hat{a}^\dagger_{s,-\boldsymbol{p}} \Big).
\end{aligned} \tag{7.92}$$

Now T_a are Hermitian and purely imaginary, therefore antisymmetric: $(\mathsf{T}_a)_{rs} = (\mathsf{T}_a)^*_{sr} = -(\mathsf{T}_a)_{sr}$. Hence

$$\int d^3p\, (\mathsf{T}_a)_{rs} \hat{a}_{r,\boldsymbol{p}} \hat{a}_{s,-\boldsymbol{p}} = - \int d^3p\, (\mathsf{T}_a)_{sr} \hat{a}_{r,-\boldsymbol{p}} \hat{a}_{s,\boldsymbol{p}} = - \int d^3p\, (\mathsf{T}_a)_{rs} \hat{a}_{r,\boldsymbol{p}} \hat{a}_{s,-\boldsymbol{p}} = 0, \tag{7.93}$$

and similarly for the term $\hat{a}^\dagger_{r,\boldsymbol{p}} \hat{a}^\dagger_{s,-\boldsymbol{p}}$. Therefore we obtain

$$\hat{Q}_a = -i \int d^3x \ :\hat{\pi}_r (\mathsf{T}_a)_{rs} \hat{\phi}_s: = \frac{1}{2} \int d^3p\, (\mathsf{T}_a)_{rs} \Big(- \hat{a}^\dagger_{s,\boldsymbol{p}} \hat{a}_{r,\boldsymbol{p}} + \hat{a}^\dagger_{r,\boldsymbol{p}} \hat{a}_{s,\boldsymbol{p}} \Big) = \int d^3p\, \hat{a}^\dagger_{r,\boldsymbol{p}} (\mathsf{T}_a)_{rs} \hat{a}_{s,\boldsymbol{p}}. \tag{7.94}$$

Exercise 32. *Algebra of Noether charges.* Show that the Noether charges

$$\hat{Q}_a = \int d^3p\, \hat{a}^\dagger_{r,\boldsymbol{p}} (\mathsf{T}_a)_{rs} \hat{a}_{s,\boldsymbol{p}} \tag{7.95}$$

satisfy the same commutation rules as the generators of internal transformations T_a.

Solution:

A straightforward application of commutation rules for creation and annihilation operators, Eq. (7.33), yields

$$\begin{aligned}
[\hat{Q}_a, \hat{Q}_b] &= \int d^3p\, d^3p'\, (\mathsf{T}_a)_{rs} (\mathsf{T}_b)_{r's'} \big[\hat{a}^\dagger_{r,\boldsymbol{p}} \hat{a}_{s,\boldsymbol{p}}\, ,\, \hat{a}^\dagger_{r',\boldsymbol{p}'} \hat{a}_{s',\boldsymbol{p}'} \big] \\
&= \int d^3p\, d^3p'\, (\mathsf{T}_a)_{rs} (\mathsf{T}_b)_{r's'} \Big(\hat{a}^\dagger_{r,\boldsymbol{p}} [\hat{a}_{s,\boldsymbol{p}}\, ,\, \hat{a}^\dagger_{r',\boldsymbol{p}'}] \hat{a}_{s',\boldsymbol{p}'} + \hat{a}^\dagger_{r',\boldsymbol{p}'} [\hat{a}^\dagger_{r,\boldsymbol{p}}\, ,\, \hat{a}_{s',\boldsymbol{p}'}] \hat{a}_{s,\boldsymbol{p}} \Big) \\
&= \int d^3p\, d^3p'\, (\mathsf{T}_a)_{rs} (\mathsf{T}_b)_{r's'} \Big(\hat{a}^\dagger_{r,\boldsymbol{p}} \hat{a}_{s',\boldsymbol{p}'}\, \delta_{sr'} \delta(\boldsymbol{p}-\boldsymbol{p}') - \hat{a}^\dagger_{r',\boldsymbol{p}'} \hat{a}_{s,\boldsymbol{p}}\, \delta_{rs'} \delta(\boldsymbol{p}-\boldsymbol{p}') \Big) \\
&= \int d^3p\, \Big((\mathsf{T}_a)_{rs} (\mathsf{T}_b)_{ss'}\, \hat{a}^\dagger_{r,\boldsymbol{p}} \hat{a}_{s',\boldsymbol{p}} - (\mathsf{T}_a)_{rs} (\mathsf{T}_b)_{r'r}\, \hat{a}^\dagger_{r',\boldsymbol{p}} \hat{a}_{s,\boldsymbol{p}} \Big) \\
&= \int d^3p\, \Big((\mathsf{T}_a \mathsf{T}_b)_{rs} - (\mathsf{T}_b \mathsf{T}_a)_{rs} \Big)\, \hat{a}^\dagger_{r,\boldsymbol{p}} \hat{a}_{s,\boldsymbol{p}}
\end{aligned} \tag{7.96}$$

If now

$$[\mathsf{T}_a, \mathsf{T}_b] = f_{abc}\mathsf{T}_c \tag{7.97}$$

for some structure constants f_{abc}, we find

$$[\hat{Q}_a, \hat{Q}_b] = \int d^3p \, f_{abc}(\mathsf{T}_c)_{rs} \, \hat{a}^\dagger_{r,\boldsymbol{p}}\hat{a}_{s,\boldsymbol{p}} = f_{abc}\hat{Q}_c. \tag{7.98}$$

Chapter 8

Canonical quantization of Dirac field

When quantizing the Klein-Gordon field we repeatedly used the intuition (as well as concrete formulas) gained while studying quantum oscillatory systems in Section 6.1. However, this approach, although intuitively pleasing, has its limits — not every field theory has a direct interpretation as a continuum of coupled oscillators. (Recall the non-relativistic 'Schrödinger' field of Section 6.2 whose Lagrangian contains the term $\psi^* \partial_t \psi$, which has no analogue in the standard oscillatory Lagrangian (5.6).) This is not really an issue since what we are ultimately interested in is a general solution of the quantum field's equation of motion in the form of a mode expansion, where we can identify spatial shapes of the modes, energies, and polarization states.

This will be our strategy in dealing with the Dirac field described by the Lagrangian (studied in Exercise 22)

$$\mathscr{L}(\Psi, \bar{\Psi}, \partial_\mu \Psi, \partial_\mu \bar{\Psi}) = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi = \Psi^\dagger (i\partial_t - H_D)\Psi, \quad \text{where} \quad H_D = -i\gamma^0\gamma^i\partial_i + m\gamma^0 \tag{8.1}$$

is the quantum-mechanical Dirac Hamiltonian introduced in Eq. (4.5). (We omit the 'hat' over H_D to reserve it for quantum field operators.) The four-component fields $\Psi = (\psi_\alpha)$ and $\bar{\Psi} = (\bar{\psi}_\alpha)$ are treated as independent. From the Dirac Lagrangian we easily deduce the canonical momenta

$$\pi_\alpha = \frac{\partial \mathscr{L}}{\partial(\partial_0 \psi_\alpha)} = i\psi^*_\alpha \quad , \quad \bar{\pi}_\alpha = \frac{\partial \mathscr{L}}{\partial(\partial_0 \bar{\psi}_\alpha)} = 0, \tag{8.2}$$

and the total Hamiltonian

$$H = \int d^3x\, (\pi_\alpha \dot{\psi}_\alpha - \mathscr{L}) = \int d^3x\, (i\Psi^\dagger \partial_t \Psi - \mathscr{L}) = \int d^3x\, \Psi^\dagger H_D \Psi. \tag{8.3}$$

Thus, there are only eight independent canonical fields ψ_α and π_α ($\bar{\pi}_\alpha$ vanish and $\bar{\psi}_\alpha$ are algebraic combinations of the momenta π_α).

The Dirac Lagrangian (8.1) yields the Dirac equation $(i\gamma^\mu\partial_\mu - m)\Psi = 0$, which describes, according to Chapter 3, particles with spin $\frac{1}{2}$ (such as electrons). These are fermions, and so the corresponding quantum field theory should implement the Pauli exclusion principle. For the quantized canonical fields $\hat{\psi}_\alpha$ and $\hat{\pi}_\alpha = i\hat{\psi}^\dagger_\alpha$ we therefore postulate the equal-time anticommutation relations (as we have seen in the non-relativistic case in Section (6.2.2))

$$\{\hat{\psi}_\alpha(\boldsymbol{x}, t), \hat{\pi}_\beta(\boldsymbol{y}, t)\} = i\,\delta_{\alpha\beta}\,\delta(\boldsymbol{x} - \boldsymbol{y}) \quad , \quad \{\hat{\psi}_\alpha(\boldsymbol{x}, t), \hat{\psi}_\beta(\boldsymbol{y}, t)\} = \{\hat{\pi}_\alpha(\boldsymbol{x}, t), \hat{\pi}_\beta(\boldsymbol{y}, t)\} = 0. \tag{8.4}$$

Further reason for using anticommutators (rather than commutators) is to have the total energy bounded from below, as will be discussed later (Eq. (8.15)).

The Heisenberg equation of motion for the quantum field $\hat{\Psi}(x)$ reads

$$\partial_t \hat{\psi}_\alpha(\boldsymbol{x}, t) = -i[\hat{\psi}_\alpha(\boldsymbol{x}, t), \hat{H}] = -\int d^3x' \, [\hat{\psi}_\alpha(\boldsymbol{x}, t), \hat{\pi}_\beta(\boldsymbol{x}', t)(H'_D)_{\beta\kappa}\hat{\psi}_\kappa(\boldsymbol{x}', t)], \tag{8.5}$$

where $(H'_D)_{\beta\kappa} = -i(\gamma^0\gamma^i)_{\beta\kappa}\partial'_i + m(\gamma^0)_{\beta\kappa}$, and using the anticommutator Leibniz rule $[A, BC] = \{A, B\}C - B\{A, C\}$ we find

$$\partial_t \hat{\psi}_\alpha(\boldsymbol{x}, t) = -\int d^3x' \, i\, \delta_{\alpha\beta}\, \delta(\boldsymbol{x} - \boldsymbol{x}')(H'_D)_{\beta\kappa}\hat{\psi}_\kappa(\boldsymbol{x}', t) = -i(H_D\hat{\psi})_\alpha(\boldsymbol{x}, t) \quad \to \quad i\partial_t \hat{\Psi} = H_D \hat{\Psi}. \tag{8.6}$$

The same conclusion would, in fact, be reached had we used commutators instead of anticommutators in Eq. (8.4); in either case we obtain the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\hat{\Psi} = 0 \tag{8.7}$$

for the quantum field $\hat{\Psi}(x)$.

8.1 Mode expansion of Dirac field

We can copy a general solution of the quantum-field-theoretic Dirac equation (8.7) from Chapter 3, Eq. (3.29) (only replacing the classical plane wave amplitudes $B_{\boldsymbol{p},s}$, $D^*_{\boldsymbol{p},s}$ by constant operators $\hat{b}_{\boldsymbol{p},s}$, $\hat{d}^\dagger_{\boldsymbol{p},s}$). Thus we obtain a mode expansion of the quantized Dirac field

$$\hat{\Psi}(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega_{\boldsymbol{p}}}} \Big(\hat{b}_{\boldsymbol{p},s} u(\boldsymbol{p}, s) e^{-ip\cdot x} + \hat{d}^\dagger_{\boldsymbol{p},s} v(\boldsymbol{p}, s) e^{ip\cdot x} \Big), \quad \text{where} \quad p_0 = \omega_{\boldsymbol{p}} \tag{8.8}$$

and its Dirac conjugate

$$\hat{\bar{\Psi}}(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega_{\boldsymbol{p}}}} \Big(\hat{d}_{\boldsymbol{p},s} \bar{v}(\boldsymbol{p}, s) e^{-ip\cdot x} + \hat{b}^\dagger_{\boldsymbol{p},s} \bar{u}(\boldsymbol{p}, s) e^{ip\cdot x} \Big). \tag{8.9}$$

The polarisation spinors u and v satisfy the algebraic equations

$$(\gamma^\mu p_\mu - m)u(\boldsymbol{p}) = 0 \quad , \quad (\gamma^\mu p_\mu + m)v(\boldsymbol{p}) = 0, \tag{8.10}$$

whose solutions are given by (standard Dirac representation of γ-matrices assumed)

$$u(\boldsymbol{p}, s) = \frac{\gamma^\mu p_\mu + m}{\sqrt{2m(p_0 + m)}} u(\boldsymbol{0}, s) \quad , \quad v(\boldsymbol{p}, s) = \frac{-\gamma^\mu p_\mu + m}{\sqrt{2m(p_0 + m)}} v(\boldsymbol{0}, s), \tag{8.11}$$

with

$$u(\boldsymbol{0}, \tfrac{1}{2}) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad u(\boldsymbol{0}, -\tfrac{1}{2}) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad v(\boldsymbol{0}, \tfrac{1}{2}) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad v(\boldsymbol{0}, -\tfrac{1}{2}) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{8.12}$$

Note that here, unlike in the case of multicomponent Klein-Gordon field, the label s refers to polarization states (spin 'up' for $s = \frac{1}{2}$ and spin 'down' for $s = -\frac{1}{2}$); it does not enumerate the components of the field Ψ (this is the role of indices $\alpha, \beta, \ldots$). We do not sum over s unless $\sum_s$ is explicitly displayed.

The amplitude operators $\hat{b}_{\boldsymbol{p},s}, \hat{b}^\dagger_{\boldsymbol{p},s}$ and $\hat{d}_{\boldsymbol{p},s}\hat{d}^\dagger_{\boldsymbol{p},s}$ can be expressed in terms of $\hat{\Psi}$ and $\hat{\bar{\Psi}}$, and in Exercise 34 we find the following anticommutation relations:

$$\{\hat{b}_{\boldsymbol{p},s}, \hat{b}^\dagger_{\boldsymbol{p}',s'}\} = \delta_{ss'}\,\delta(\boldsymbol{p}-\boldsymbol{p}') \quad , \quad \{\hat{d}_{\boldsymbol{p},s}, \hat{d}^\dagger_{\boldsymbol{p}',s'}\} = \delta_{ss'}\,\delta(\boldsymbol{p}-\boldsymbol{p}'), \tag{8.13}$$

whereas anticommutators involving the remaining combinations of the operators $\hat{b}, \hat{b}^\dagger, \hat{d}, \hat{d}^\dagger$ vanish:

$$\{\hat{b}, \hat{b}\} = \{\hat{b}^\dagger, \hat{b}^\dagger\} = \{\hat{d}, \hat{d}\} = \{\hat{d}^\dagger, \hat{d}^\dagger\} = \{\hat{b}, \hat{d}\} = \{\hat{b}, \hat{d}^\dagger\} = \{\hat{b}^\dagger, \hat{d}\} = \{\hat{b}^\dagger, \hat{d}^\dagger\} = 0 \tag{8.14}$$

(Here we have, for short, hidden the arbitrary parameters $\boldsymbol{p}, \boldsymbol{p}'$ and s, s'.) The Fock space corresponding to the quantized Dirac field can be built, similarly as in Section 6.2.2, by acting with fermionic creation operators $\hat{b}^\dagger_{\boldsymbol{p},s}$, $\hat{d}^\dagger_{\boldsymbol{p},s}$ on a vacuum state $|0\rangle$, which gets annihilated by all the operators $\hat{b}_{\boldsymbol{p},s}, \hat{d}_{\boldsymbol{p},s}$.

Plugging this mode expansion into the (quantized) Hamiltonian operator (8.3), we find, in Exercise 33, the expression

$$\hat{H} = \sum_s \int d^3p\,\omega_{\boldsymbol{p}} \Big(\hat{b}^\dagger_{\boldsymbol{p},s}\hat{b}_{\boldsymbol{p},s} - \hat{d}_{\boldsymbol{p},s}\hat{d}^\dagger_{\boldsymbol{p},s}\Big) = \sum_s \int d^3p\,\omega_{\boldsymbol{p}} \Big(\hat{b}^\dagger_{\boldsymbol{p},s}\hat{b}_{\boldsymbol{p},s} + \hat{d}^\dagger_{\boldsymbol{p},s}\hat{d}_{\boldsymbol{p},s} - \delta(\boldsymbol{0})\Big). \tag{8.15}$$

where in the second step we reordered the d and $d^\dagger$ operators using the anticommutation relation (8.13). Subtracting the infinite vacuum energy (note that this time the sign is opposite to the bosonic case in Eq. (7.17)) we obtain the normal-ordered Hamiltonian

$$:\hat{H}: = \hat{H} - \langle 0|\,\hat{H}\,|0\rangle = \sum_s \int d^3p\,\omega_{\boldsymbol{p}} \Big(\hat{b}^\dagger_{\boldsymbol{p},s}\hat{b}_{\boldsymbol{p},s} + \hat{d}^\dagger_{\boldsymbol{p},s}\hat{d}_{\boldsymbol{p},s}\Big), \tag{8.16}$$

which is positive-definite on all states in the Fock space. For fermionic operators the normal ordering $:\ldots:$ is defined slightly differently compared to the bosonic case. It again pushes all creation operators to the left of all annihilation operators, but every swap in this process incurs a factor of -1 (for example, $:d\,d^\dagger b\,b^\dagger: = (-1)^3 d^\dagger b^\dagger d\,b$). Under normal ordering, fermionic operators behave as Grassmann symbols (see Eq. (6.108)). With this rule, the form (8.16) follows from the first expression in Eq. (8.15).

The use of anticommutators now turns out as crucial, for if we postulated *commutation* relations between the fields in Eq. (8.4), and hence obtained *commutation* relations for the creation and annihilation operators in Eq. (8.13), the term $\hat{d}^\dagger_{\boldsymbol{p},s}\hat{d}_{\boldsymbol{p},s}$ would keep the negative sign. As a result, the field would indefinitely create more and more d-type particles while minimizing its total energy. Therefore, the use of anticommutators for the Dirac field is dictated by the form of the Dirac Lagrangian (8.1).

When developing his relativistic quantum mechanics for spin-$\frac{1}{2}$ particles, Dirac was concerned with negative-energy states (those with wave-functions $v(\boldsymbol{p})e^{ip\cdot x}$). He postulated what is now known as the 'Dirac sea': that the ground state has all negative-energy levels filled with particles (this is possible due to Pauli exclusion principle for fermions), but it is homogeneous and hence unobservable. Occasionally a negative-energy electron in the Dirac sea can absorb a photon of large enough energy in order to overcome the energy gap of $2mc^2$, and jump to one of the positive-energy states. This creates a hole, which Dirac interpreted as an antiparticle — the positron (discovered a few years later in 1932).

We have seen that on the level of quantum field theory the problem with negative-energy states is avoided by adopting *anticommutation* relations between the field operators. Particles created by the operators $d^\dagger$ will shortly turn out to be antiparticles of the particles created by $b^\dagger$.

8.2 States and conserved quantities

We will show that the states created from the vacuum by operators $\hat{b}^\dagger_{\boldsymbol{p},s}$, $\hat{d}^\dagger_{\boldsymbol{p},s}$ are eigenstates of the operators of total four-momentum $\hat{P}_\mu$ and total charge $\hat{Q}$ of the Dirac field (analogously to the case of complex Klein-Gordon field, Section 7.2.1).

First, let us recall Exercise 22 where the energy-momentum tensor of the Dirac field and the Dirac current were determined:

$$T^\mu{}_\nu = i\bar{\Psi}\gamma^\mu\partial_\nu\Psi - \delta^\mu_\nu\mathscr{L} \quad , \quad J^\mu = \bar{\Psi}\gamma^\mu\Psi. \tag{8.17}$$

The total four-momentum operator $\hat{P}_\mu$ and the total charge operator $\hat{Q}$ are then given by

$$\hat{P}_0 = :\hat{H}: \quad , \quad \hat{P}_j = \int d^3x \; :\hat{T}^0{}_j: = \int d^3x \; :i\hat{\Psi}^\dagger\partial_j\hat{\Psi}: \quad , \quad \hat{Q} = \int d^3x \; :J^0: = \int d^3x \; :\hat{\Psi}^\dagger\hat{\Psi}:, \tag{8.18}$$

where fermionic normal ordering has been used. Plugging in the mode expansions (8.8) and (8.9), and using the anticommutation relations (8.13) and (8.14), they read (by calculations analogous to those in Exercise 33)

$$\hat{P}_\mu = \sum_s \int d^3p\, p_\mu \Big(\hat{b}^\dagger_{\boldsymbol{p},s}\hat{b}_{\boldsymbol{p},s} + \hat{d}^\dagger_{\boldsymbol{p},s}\hat{d}_{\boldsymbol{p},s}\Big) \quad , \quad \hat{Q} = \sum_s \int d^3p \,\Big(\hat{b}^\dagger_{\boldsymbol{p},s}\hat{b}_{\boldsymbol{p},s} - \hat{d}^\dagger_{\boldsymbol{p},s}\hat{d}_{\boldsymbol{p},s}\Big). \tag{8.19}$$

The 'anticommutator Leibniz rule' $[AB, C] = A\{B, C\} - \{A, C\}B$ then yields

$$\begin{aligned}[\hat{P}_\mu, \hat{b}^\dagger_{\boldsymbol{p},s}] &= \sum_{s'} \int d^3p'\, p'_\mu \Big[\hat{b}^\dagger_{\boldsymbol{p}',s'}\hat{b}_{\boldsymbol{p}',s'} + \hat{d}^\dagger_{\boldsymbol{p}',s'}\hat{d}_{\boldsymbol{p}',s'} \,,\, \hat{b}^\dagger_{\boldsymbol{p},s}\Big] \\ &= \sum_{s'} \int d^3p'\, p'_\mu\, \hat{b}^\dagger_{\boldsymbol{p}',s'}\Big\{\hat{b}_{\boldsymbol{p}',s'}, \hat{b}^\dagger_{\boldsymbol{p},s}\Big\} \\ &= p_\mu \hat{b}^\dagger_{\boldsymbol{p},s}. \end{aligned} \tag{8.20}$$

By similar calculations, and Hermitian conjugation, we find altogether

$$[\hat{P}_\mu, \hat{b}^\dagger_{\boldsymbol{p},s}] = p_\mu\hat{b}^\dagger_{\boldsymbol{p},s} \quad , \quad [\hat{P}_\mu, \hat{b}_{\boldsymbol{p},s}] = -p_\mu\hat{b}_{\boldsymbol{p},s} \quad , \quad [\hat{P}_\mu, \hat{d}^\dagger_{\boldsymbol{p},s}] = p_\mu\hat{d}^\dagger_{\boldsymbol{p},s} \quad , \quad [\hat{P}_\mu, \hat{d}_{\boldsymbol{p},s}] = -p_\mu\hat{d}_{\boldsymbol{p},s}, \tag{8.21}$$

and

$$[\hat{Q}, \hat{b}^\dagger_{\boldsymbol{p},s}] = \hat{b}^\dagger_{\boldsymbol{p},s} \quad , \quad [\hat{Q}, \hat{b}_{\boldsymbol{p},s}] = -\hat{b}_{\boldsymbol{p},s} \quad , \quad [\hat{Q}, \hat{d}^\dagger_{\boldsymbol{p},s}] = -\hat{d}^\dagger_{\boldsymbol{p},s} \quad , \quad [\hat{Q}, \hat{d}_{\boldsymbol{p},s}] = \hat{d}_{\boldsymbol{p},s}. \tag{8.22}$$

This is what we need in order to evaluate the action of the operators $\hat{P}_\mu$ and $\hat{Q}$ on a generic state

$$|\alpha\rangle \equiv \hat{b}^\dagger_{\boldsymbol{p}_1,s_1}\cdots\hat{b}^\dagger_{\boldsymbol{p}_{n_b},s_{n_b}}\hat{d}^\dagger_{\boldsymbol{p}'_1,s'_1}\cdots\hat{d}^\dagger_{\boldsymbol{p}'_{n_d},s'_{n_d}}|0\rangle\,. \tag{8.23}$$

In complete analogy with Eqs. (7.49) and (7.55) for the complex Klein-Gordon field we find

$$\begin{aligned}\hat{P}^\mu|\alpha\rangle &= \Big(\sum_{\ell=1}^{n_b} p^\mu_\ell + \sum_{\ell'=1}^{n_d} p^\mu_{\ell'}\Big)|\alpha\rangle\,, \\ \hat{Q}\,|\alpha\rangle &= (n_b - n_d)\,|\alpha\rangle\,. \end{aligned} \tag{8.24}$$

In words, the operator $\hat{b}^\dagger_{\boldsymbol{p},s}$ creates a particle with four-momentum p_μ, spin projection s, and charge $+1$; the operator $\hat{d}^\dagger_{\boldsymbol{p},s}$ creates an antiparticle with four-momentum p_μ, spin projection s, and charge -1. (Note that there is no conserved spin operator. Only spin *plus* orbital angular momentum yield conserved Noether charges — see Eq. (5.71).)

Let us also mention that the commutator identities (8.21) and (8.22) imply for the Dirac field the formulas

$$[ia^\mu \hat{P}_\mu, \hat{\Psi}(x)] = a^\mu \partial_\mu \hat{\Psi}(x) \quad \rightarrow \quad e^{ia^\mu \hat{P}_\mu} \hat{\Psi}(x) e^{-ia^\mu \hat{P}_\mu} = \hat{\Psi}(x+a), \tag{8.25}$$

and

$$[\hat{Q}, \hat{\Psi}(x)] = -\hat{\Psi}(x) \quad \rightarrow \quad e^{i\lambda \hat{Q}} \hat{\Psi}(x) e^{-i\lambda \hat{Q}} = e^{-i\lambda} \hat{\Psi}(x), \tag{8.26}$$

analogous to Klein-Gordon field's Eqs. (7.52) and (7.54).

Finally, relativistically normalised one-particle states are defined

$$|\boldsymbol{p}, s\rangle \equiv \frac{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}}{\sqrt{2m}} \hat{b}^\dagger_{\boldsymbol{p},s} |0\rangle \quad , \quad |\bar{\boldsymbol{p}}, \bar{s}\rangle \equiv \frac{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}}{\sqrt{2m}} \hat{d}^\dagger_{\boldsymbol{p},s} |0\rangle\,, \tag{8.27}$$

with

$$\langle \boldsymbol{p}, s|\boldsymbol{p}', s'\rangle = \langle \bar{\boldsymbol{p}}, s|\bar{\boldsymbol{p}}', s'\rangle = (2\pi)^3 \frac{\omega_{\boldsymbol{p}}}{m} \delta_{s,s'} \delta(\boldsymbol{p}-\boldsymbol{p}'), \tag{8.28}$$

where the extra factor $\frac{1}{\sqrt{2m}}$ (compare to the scalar field case, Section 7.3.1) has been added to obtain simple expressions

$$\langle 0| \hat{\Psi}(x) |\boldsymbol{p}, s\rangle = u(\boldsymbol{p}, s)\, e^{-ip\cdot x} \quad , \quad \langle 0| \hat{\bar{\Psi}}(x) |\bar{\boldsymbol{p}}, \bar{s}\rangle = \bar{v}(\boldsymbol{p}, s)\, e^{-ip\cdot x}. \tag{8.29}$$

Since $(i\gamma^\mu \partial_\mu - m)\hat{\Psi} = 0$, and hence also $i\partial_\mu \hat{\bar{\Psi}} \gamma^\mu + m\hat{\bar{\Psi}} = 0$, these one-particle wave functions satisfy the Dirac equation, and its Dirac conjugate, respectively.

8.3 Exercises

Exercise 33. *Dirac field Hamiltonian in terms of creation and annihilation operators.* Using the mode expansion of the Dirac field,

$$\hat{\Psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega_{\boldsymbol{p}}}} \left(\hat{b}_{\boldsymbol{p},s} u(\boldsymbol{p}, s) e^{-ip\cdot x} + \hat{d}^\dagger_{\boldsymbol{p},s} v(\boldsymbol{p}, s) e^{ip\cdot x} \right) \tag{8.30}$$

derive the formula (8.15) for the Hamiltonian operator of the Dirac field.

Solution:

Since $\hat{\Psi}(x)$ satisfies the Dirac equation, $(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\hat{\Psi} = 0$, we can cast the Hamiltonian as

$$\hat{H} = \int d^3x\, \hat{\bar{\Psi}}(-i\gamma^i\partial_i + m)\hat{\Psi} = \int d^3x\, \hat{\Psi}^\dagger i\partial_0 \hat{\Psi} \tag{8.31}$$

We shall need the following versions of the mode expansion:

$$\begin{aligned}
\hat{\Psi}^\dagger(\boldsymbol{x}, 0) &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega_{\boldsymbol{p}}}} \left(\hat{d}_{\boldsymbol{p},s} v^\dagger(\boldsymbol{p}, s) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{b}^\dagger_{\boldsymbol{p},s} u^\dagger(\boldsymbol{p}, s) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \right) \\
i\partial_0\hat{\Psi}(\boldsymbol{x}, 0) &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega_{\boldsymbol{p}}}} \left(\omega_{\boldsymbol{p}}\, \hat{b}_{\boldsymbol{p},s} u(\boldsymbol{p}, s) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - \omega_{\boldsymbol{p}}\, \hat{d}^\dagger_{\boldsymbol{p},s} v(\boldsymbol{p}, s) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \right)
\end{aligned} \tag{8.32}$$

where we have take the liberty to choose (as in Exercise 29) time $t = 0$, since $\hat{H}$ is time-independent. Now calculate

$$\begin{aligned}
\hat{H} &= \sum_{s,s'} \int \frac{d^3x\, d^3p\, d^3p'}{(2\pi)^3 \sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'}}} m\, \omega_{\boldsymbol{p}'} \\
&\quad \times \left(\hat{d}_{\boldsymbol{p},s} v^\dagger(\boldsymbol{p}, s) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{b}^\dagger_{\boldsymbol{p},s} u^\dagger(\boldsymbol{p}, s) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \right) \left(\hat{b}_{\boldsymbol{p}',s'} u(\boldsymbol{p}', s') e^{i\boldsymbol{p}'\cdot\boldsymbol{x}} - \hat{d}^\dagger_{\boldsymbol{p}',s'} v(\boldsymbol{p}', s') e^{-i\boldsymbol{p}'\cdot\boldsymbol{x}} \right) \\
&= \sum_{s,s'} \int \frac{d^3p\, d^3p'}{\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'}}} m\omega_{\boldsymbol{p}'} \Big(\hat{d}_{\boldsymbol{p},s}\hat{b}_{\boldsymbol{p}',s'} v^\dagger(\boldsymbol{p}, s) u(\boldsymbol{p}', s') \delta(\boldsymbol{p}+\boldsymbol{p}') - \hat{d}_{\boldsymbol{p},s}\hat{d}^\dagger_{\boldsymbol{p}',s'} v^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}', s') \delta(\boldsymbol{p}-\boldsymbol{p}') \\
&\qquad + \hat{b}^\dagger_{\boldsymbol{p},s}\hat{b}_{\boldsymbol{p}',s'} u^\dagger(\boldsymbol{p}, s) u(\boldsymbol{p}', s') \delta(-\boldsymbol{p}+\boldsymbol{p}') - \hat{b}^\dagger_{\boldsymbol{p},s}\hat{d}^\dagger_{\boldsymbol{p}',s'} u^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}', s') \delta(-\boldsymbol{p}-\boldsymbol{p}) \Big) \\
&= \sum_{s,s'} \int d^3p\, m \Big(\hat{d}_{\boldsymbol{p},s}\hat{b}_{-\boldsymbol{p},s'} v^\dagger(\boldsymbol{p}, s) u(-\boldsymbol{p}, s') - \hat{d}_{\boldsymbol{p},s}\hat{d}^\dagger_{\boldsymbol{p},s'} v^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}, s') \\
&\qquad + \hat{b}^\dagger_{\boldsymbol{p},s}\hat{b}_{\boldsymbol{p},s'} u^\dagger(\boldsymbol{p}, s) u(\boldsymbol{p}, s') - \hat{b}^\dagger_{\boldsymbol{p},s}\hat{d}^\dagger_{-\boldsymbol{p},s'} u^\dagger(\boldsymbol{p}, s) v(-\boldsymbol{p}, s') \Big)
\end{aligned} \tag{8.33}$$

Using the identities for polarisation spinors from Eq. (3.31),

$$u^\dagger(\boldsymbol{p}, s) u(\boldsymbol{p}, s') = \frac{\omega_{\boldsymbol{p}}}{m}\delta_{ss'} \quad , \quad v^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}, s') = \frac{\omega_{\boldsymbol{p}}}{m}\delta_{ss'} \quad , \quad u^\dagger(\boldsymbol{p}, s) v(-\boldsymbol{p}, s') = 0, \tag{8.34}$$

we finally obtain

$$\hat{H} = \sum_{s,s'} \int d^3p\, m \left(-\frac{\omega_{\boldsymbol{p}}}{m}\delta_{ss'} \hat{d}_{\boldsymbol{p},s}\hat{d}^\dagger_{\boldsymbol{p},s'} + \frac{\omega_{\boldsymbol{p}}}{m}\delta_{ss'} \hat{b}^\dagger_{\boldsymbol{p},s}\hat{b}_{\boldsymbol{p},s'} \right) = \sum_s \int d^3p\, \omega_{\boldsymbol{p}} \left(\hat{b}^\dagger_{\boldsymbol{p},s}\hat{b}_{\boldsymbol{p},s} - \hat{d}_{\boldsymbol{p},s}\hat{d}^\dagger_{\boldsymbol{p},s} \right). \tag{8.35}$$

Exercise 34. *Anticommutation relations for creation and annihilation operators.* Using the canonical anticommutation relations

$$\{\hat{\psi}_\alpha(\boldsymbol{x},t), \hat{\pi}_\beta(\boldsymbol{y},t)\} = i\,\delta_{\alpha\beta}\,\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{8.36}$$

derive the anticommutation relations between creation and annihilation operators,

$$\{\hat{b}_{\boldsymbol{p},s}, \hat{b}^\dagger_{\boldsymbol{p}',s'}\} = \delta_{ss'}\,\delta(\boldsymbol{p}-\boldsymbol{p}') \quad , \quad \{\hat{d}_{\boldsymbol{p},s}, \hat{d}^\dagger_{\boldsymbol{p}',s'}\} = \delta_{ss'}\,\delta(\boldsymbol{p}-\boldsymbol{p}'). \tag{8.37}$$

Solution:

Taking the Fourier transform of the mode expansion (8.30)

$$\begin{aligned}\int d^3x\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\hat{\Psi}(\boldsymbol{x},0) &= \sum_{s'}\int \frac{d^3p'}{(2\pi)^{3/2}}\sqrt{\frac{m}{\omega_{\boldsymbol{p}'}}}(2\pi)^3\Big(\hat{b}_{\boldsymbol{p}',s'}u(\boldsymbol{p}',s')\delta(\boldsymbol{p}-\boldsymbol{p}') + \hat{d}^\dagger_{\boldsymbol{p}',s'}v(\boldsymbol{p}',s')\delta(\boldsymbol{p}+\boldsymbol{p}')\Big)\\ &= \sum_{s'}\sqrt{\frac{m}{\omega_{\boldsymbol{p}}}}(2\pi)^{3/2}\Big(\hat{b}_{\boldsymbol{p},s'}u(\boldsymbol{p},s') + \hat{d}^\dagger_{-\boldsymbol{p},s'}v(-\boldsymbol{p},s')\Big),\end{aligned} \tag{8.38}$$

and projecting by $u^\dagger(\boldsymbol{p},s)$ with a help of the identities (8.34)

$$u^\dagger(\boldsymbol{p},s)\int d^3x\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\hat{\Psi}(\boldsymbol{x},0) = \sum_{s'}\sqrt{\frac{m}{\omega_{\boldsymbol{p}}}}(2\pi)^{3/2}\hat{b}_{\boldsymbol{p},s'}\frac{\omega_{\boldsymbol{p}}}{m}\delta_{ss'} = (2\pi)^{3/2}\hat{b}_{\boldsymbol{p},s}\sqrt{\frac{\omega_{\boldsymbol{p}}}{m}} \tag{8.39}$$

we obtain the annihilation operator $\hat{b}_{\boldsymbol{p},s}$ expressed in terms of the field $\hat{\Psi}$,

$$\hat{b}_{\boldsymbol{p},s} = \int \frac{d^3x}{(2\pi)^{3/2}}\sqrt{\frac{m}{\omega_{\boldsymbol{p}}}}\,u^\dagger(\boldsymbol{p},s)\hat{\Psi}(\boldsymbol{x},0)\,e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}. \tag{8.40}$$

Now we can calculate

$$\{\hat{b}_{\boldsymbol{p},s}, \hat{b}^\dagger_{\boldsymbol{p}',s'}\} = \int \frac{d^3x\,d^3x'}{(2\pi)^3}\frac{m}{\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'}}}\,\{u^\dagger(\boldsymbol{p},s)\hat{\Psi}(\boldsymbol{x},0), \hat{\Psi}^\dagger(\boldsymbol{x}',0)u(\boldsymbol{p}',s')\}\,e^{-i\boldsymbol{p}\cdot\boldsymbol{x}+i\boldsymbol{p}'\cdot\boldsymbol{x}'}, \tag{8.41}$$

where the anticommutator reads

$$u^*_\alpha(\boldsymbol{p},s)\{\hat{\psi}_\alpha(\boldsymbol{x},0), \hat{\psi}^\dagger_\beta(\boldsymbol{x}',0)\}u_\beta(\boldsymbol{p}',s') = u^*_\alpha(\boldsymbol{p},s)\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')u_\beta(\boldsymbol{p}',s') = u^\dagger(\boldsymbol{p},s)u(\boldsymbol{p}',s')\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{8.42}$$

This allows to eliminate the integrations over $\boldsymbol{x}$ and $\boldsymbol{x}'$,

$$\{\hat{b}_{\boldsymbol{p},s}, \hat{b}^\dagger_{\boldsymbol{p}',s'}\} = \int \frac{d^3x}{(2\pi)^3}\frac{m}{\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'}}}\,u^\dagger(\boldsymbol{p},s)u(\boldsymbol{p}',s')\,e^{i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{x}} = \frac{m}{\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'}}}\,u^\dagger(\boldsymbol{p},s)u(\boldsymbol{p}',s')\delta(\boldsymbol{p}-\boldsymbol{p}'), \tag{8.43}$$

and noting that due to the delta function $\delta(\boldsymbol{p}-\boldsymbol{p}')$, $\boldsymbol{p}'$ gets identified with $\boldsymbol{p}$ in the surrounding factors, we finally arrive at

$$\{\hat{b}_{\boldsymbol{p},s}, \hat{b}^\dagger_{\boldsymbol{p}',s'}\} = \frac{m}{\omega_{\boldsymbol{p}}}u^\dagger(\boldsymbol{p},s)u(\boldsymbol{p},s')\delta(\boldsymbol{p}-\boldsymbol{p}') = \delta_{ss'}\,\delta(\boldsymbol{p}-\boldsymbol{p}'), \tag{8.44}$$

where Eq. (8.34) has been used once again.

To find the anticommutator between d and $d^\dagger$ we project Eq. (8.38) on $v^\dagger(-\boldsymbol{p},s)$,

$$v^\dagger(-\boldsymbol{p},s)\int d^3x\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\hat{\Psi}(\boldsymbol{x},0) = \sum_{s'}\sqrt{\frac{m}{\omega_{\boldsymbol{p}}}}(2\pi)^{3/2}\hat{d}^\dagger_{-\boldsymbol{p},s'}\frac{\omega_{\boldsymbol{p}}}{m}\delta_{ss'} = (2\pi)^{3/2}\hat{d}^\dagger_{-\boldsymbol{p},s}\sqrt{\frac{\omega_{\boldsymbol{p}}}{m}}, \tag{8.45}$$

and express

$$\hat{d}^{\dagger}_{\boldsymbol{p},s} = \int \frac{d^3x}{(2\pi)^{3/2}} \sqrt{\frac{m}{\omega_{\boldsymbol{p}}}}\, v^{\dagger}(\boldsymbol{p},s)\hat{\Psi}(\boldsymbol{x},0)\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}}. \tag{8.46}$$

The rest is completely analogous to the $\hat{b}$, $\hat{b}^{\dagger}$ case.

Exercise 35. *******

Chapter 9

Free-field propagators

9.1 Commutation function

Up to now our discussion of field quantization has been based on canonical commutation or anticommutation relations at equal times. In the simplest case of a real one-component Klein-Gordon field with Lagrangian (7.1) these can be stated as (the canonical momentum field $\hat{\pi} = \partial_0 \hat{\phi}$)

$$x^0 = y^0 : \quad [\hat{\phi}(x), \hat{\phi}(y)] = 0 \quad , \quad [\partial_0 \hat{\phi}(x), \hat{\phi}(y)] = -i\delta(\boldsymbol{x} - \boldsymbol{y}), \tag{9.1}$$

where the field operator is given by the mode expansion (7.14):

$$\hat{\phi}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}} \Big(\hat{a}_{\boldsymbol{p}}\, e^{-ip\cdot x} + \hat{a}^\dagger_{\boldsymbol{p}}\, e^{ip\cdot x} \Big), \quad \text{where} \quad p_0 = \omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}. \tag{9.2}$$

It would be gratifying to have a formulation of commutation relations that is relativistically invariant, i.e., that does not single out the time coordinate. For that purpose let us define the (*Pauli-Jordan*) *commutation function*

$$i\Delta(x - y) = [\hat{\phi}(x), \hat{\phi}(y)], \tag{9.3}$$

where we have anticipated that Δ is a number-valued function that only depends on the difference $x - y$. Indeed, substituting the mode expansion (9.2) we find

$$\begin{aligned}
[\hat{\phi}(x), \hat{\phi}(y)] &= \int \frac{d^3p\, d^3p'}{(2\pi)^3 \sqrt{2\omega_{\boldsymbol{p}} 2\omega_{\boldsymbol{p}'}}} \Big(\underbrace{[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p}'}]}_{\delta(\boldsymbol{p}-\boldsymbol{p}')}\, e^{-ip\cdot x} e^{ip'\cdot y} + \underbrace{[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{p}'}]}_{-\delta(\boldsymbol{p}-\boldsymbol{p}')}\, e^{ip\cdot x} e^{-ip'\cdot y} \Big) \\
&= \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \Big(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \Big), \quad \text{where} \quad p_0 = \omega_{\boldsymbol{p}}.
\end{aligned} \tag{9.4}$$

Meanwhile we observe that since the operator $\hat{\phi}(x)$ satisfies the Klein-Gordon equation, the same holds for the commutation function (regarded as a function of x):

$$(\Box_x + m^2)\Delta(x - y) = -i[(\Box_x + m^2)\hat{\phi}(x), \hat{\phi}(y)] = 0, \tag{9.5}$$

with the initial conditions

$$\Delta|_{x^0=y^0} = 0 \quad , \quad \left.\frac{\partial \Delta}{\partial x^0}\right|_{x^0=y^0} = -\delta(\boldsymbol{x} - \boldsymbol{y}) \tag{9.6}$$

given by the equal-time commutation relations (9.1). $\Delta(x-y)$ therefore describes a wave that starts from a localised distortion ("a hammer hit" at point $\boldsymbol{y}$ and time y^0) of a uniform classical field. As a remark, note that we could as well define another commutation function $i\tilde{\Delta}(x-y) = [\hat{\phi}(x), \hat{\pi}(y)]$, that would be again a solution of the Klein-Gordon equation, now with initial conditions

$$\tilde{\Delta}|_{x^0=y^0} = \delta(\boldsymbol{x}-\boldsymbol{y}) \quad , \quad \left.\frac{\partial\tilde{\Delta}}{\partial x^0}\right|_{x^0=y^0} = 0, \tag{9.7}$$

corresponding to "pulling the field infinitely high" at point $\boldsymbol{y}$ and time y^0.

The commutation function $\Delta(x-y)$ can be rewritten in a manifestly Lorentz-invariant form. To this end we make the substitution $\boldsymbol{p} \to -\boldsymbol{p}$ in the second term of expression (9.4), and introduce an extra integration over (now independent) variable p_0:

$$\begin{aligned} i\Delta(x-y) &= \int \frac{d^4p}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \Big(\delta(p_0-\omega_{\boldsymbol{p}})e^{-ip_0(x^0-y^0)+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} - \delta(p_0+\omega_{\boldsymbol{p}})e^{-ip_0(x^0-y^0)+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\Big) \\ &= \int \frac{d^4p}{(2\pi)^3} \operatorname{sgn}(p_0)\delta(p^2-m^2)\, e^{-ip\cdot(x-y)}, \end{aligned} \tag{9.8}$$

where we have used the δ-function formula (7.72). Since $d^4p' = d^4p\,|\det \mathsf{L}| = d^4p$, the resulting integral representation is invariant under orthochronous Lorentz transformations $x' = \mathsf{L}x$ (these do not change the sign of p_0, $\operatorname{sgn}(p_0)$):

$$\Delta(x'-y') = \Delta(x-y). \tag{9.9}$$

Given any two distinct space-like separated points x and y, a Lorentz boost exists such that in the new frame of reference the times become equal: $x'^0 = y'^0$. This then implies that

$$\forall x,y : (x-y)^2 < 0 : \quad [\hat{\phi}(x), \hat{\phi}(y)] = i\Delta(x-y) = i\Delta(x'-y') = [\hat{\phi}(x'), \hat{\phi}(y')]|_{x'^0=y'^0} = 0, \tag{9.10}$$

i.e., the field operators taken at two space-like separated points commute. This property is called *microcausality.* It implies that measurements at two points that have a space-like separation, i.e., those which cannot get into contact through the transmission of light signals, do not influence each other.

For the Dirac field one can similarly study anticommutation relations $\{\hat{\psi}_\alpha(x), \hat{\psi}_\beta(y)\}$ at arbitrary times, and find that local observables, i.e., bilinear combinations $\hat{\bar{\psi}}_\alpha(x)O_{\alpha\beta}(x)\hat{\psi}_\beta(x)$, commute for space-like separated points [2, p. 138] — microcausality again holds (as it does also for all the other relativistic field theories).

9.1.1 Spin statistics connection

If we tried to postulate anticommutation relations between Klein-Gordon field's creation and annihilation operators, then microcausality (on the level of bilinear observables) would be violated for spatial distances of the order of the Compton wavelength $\frac{\hbar}{mc}$ [2, Ch. 4.4]. That is, commutation relations (which imply bosonic statistics for particle excitations) are necessary in order to obtain a 'consistent' relativistic quantum field theory with the Klein-Gordon Lagrangian. At the same time, in Section 8.1 we argued that the Dirac field must be quantized with anticommutators (which imply fermionic statistics) in order to avoid infinite negative energies.

These conclusions are part of a more general result, the *spin-statistics theorem*, first derived by Pauli: Lorentz invariance, positive energies, positive norms, and causality together imply that fields with spin $0, 1, 2, \ldots$ are quantized using commutators and the corresponding particle excitations obey Bose-Einstein statistics, while fields with spin $\frac{1}{2}, \frac{3}{2}, \ldots$ are quantized using anticommutators and their particle excitations obey Fermi-Dirac statistics.

Such distinction is not available in non-relativistic field theory, Section 6.2, where the choice between commutators and anticommutators, i.e., between symmetric (bosonic) wave-functions, and antisymmetric (fermionic) wave-functions is made 'by hand'.

Quantum-field-theoretic description of particles also naturally explains why particles of the same type, even created in different parts of the universe, are all absolutely identical and indistinguishable. They are created as excitations of one and the same quantum field.

9.2 Feynman propagator of Klein-Gordon field

An essential ingredient for the perturbative investigation of interacting quantum field theories (starting in Chapter 11) is the *Feynman propagator*. For a free one-component real Klein-Gordon field it is defined as

$$i\Delta_F(x-y) = \langle 0|\, T\big(\hat{\phi}(x)\hat{\phi}(y)\big)\, |0\rangle\,, \tag{9.11}$$

where the *time ordering* $T(.)$ reorders the operators to chronological order so that the earlier time is placed on the right:

$$T\big(\hat{\phi}(x)\hat{\phi}(y)\big) = \theta(x^0-y^0)\hat{\phi}(x)\hat{\phi}(y) + \theta(y^0-x^0)\hat{\phi}(y)\hat{\phi}(x) = \begin{cases} \hat{\phi}(x)\hat{\phi}(y), & x^0 \geq y^0 \\ \hat{\phi}(y)\hat{\phi}(x), & y^0 > x^0 \end{cases}. \tag{9.12}$$

The relevance of time ordering stems from the perturbation expansion of the Dyson operator $T\exp\left(-i\int_{t_0}^{t} dt'\, \hat{H}_I^I(t')\right)$ (the evolution operator in the interaction picture), where Feynman propagators feature as basic building blocks, as we shall see in Chapter 11.

An explicit form of the Feynman propagator is found with a help of the mode expansion (9.2). First, for $x^0 > y^0$ we have

$$\langle 0|\, \hat{\phi}(x)\hat{\phi}(y)\, |0\rangle = \int \frac{d^3p\, d^3p'}{(2\pi)^3\sqrt{2\omega_{\boldsymbol{p}}2\omega_{\boldsymbol{p}'}}} \underbrace{\langle 0|\, \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}\, |0\rangle}_{\delta(\boldsymbol{p}-\boldsymbol{p}')} e^{-ip\cdot x} e^{ip'\cdot y} = \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\, e^{-ip\cdot(x-y)}, \tag{9.13}$$

and hence, altogether,

$$i\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \Big(\theta(x^0-y^0)e^{-ip\cdot(x-y)} + \theta(y^0-x^0)e^{ip\cdot(x-y)}\Big), \quad \text{where} \quad p_0 = \omega_{\boldsymbol{p}}. \tag{9.14}$$

This also shows that $\langle 0|\, T\big(\hat{\phi}(x)\hat{\phi}(y)\big)\, |0\rangle$ is a function of the difference $x-y$, which is in fact clear on theoretical grounds from translational invariance of the vacuum state, $\hat{U}(a)\,|0\rangle = |0\rangle$, where $\hat{U}(a) \equiv e^{ia^\mu \hat{P}_\mu}$ and $\hat{P}_\mu\,|0\rangle = 0$, together with the transformation property $\hat{\phi}(x) = \hat{U}(y)\hat{\phi}(x-y)\hat{U}^\dagger(y)$ (see Eq. (7.52)):

$$\langle 0|\, T\big(\hat{\phi}(x)\hat{\phi}(y)\big)\, |0\rangle = \langle 0|\, \hat{U}(y) T\big(\hat{\phi}(x-y)\hat{U}^\dagger(y)\hat{U}(y)\hat{\phi}(0)\big)\hat{U}^\dagger(y)\, |0\rangle = \langle 0|\, T\big(\hat{\phi}(x-y)\hat{\phi}(0)\big)\, |0\rangle\,. \tag{9.15}$$

A very useful (Lorentz invariant) form of the Feynman propagator can be derived by employing the integral representation of the Heaviside step function from Exercise 36,

$$\theta(x^0) = \lim_{\varepsilon\to 0_+} i\int_{-\infty}^{+\infty} \frac{dp_0}{2\pi}\, \frac{e^{-ip_0x^0}}{p_0+i\varepsilon} \quad , \quad \theta(-x^0) = \lim_{\varepsilon\to 0_+} (-i)\int_{-\infty}^{+\infty} \frac{dp_0}{2\pi}\, \frac{e^{-ip_0x^0}}{p_0-i\varepsilon}. \tag{9.16}$$

(For brevity, the limit in ε is usually not displayed, but remains implicit in all calculations.) Putting $y = 0$ this gives

$$i\Delta_F(x) = i \int \frac{d^3p\, dp_0}{(2\pi)^4 2\omega_{\boldsymbol{p}}} \left(\frac{e^{-i(p_0+\omega_{\boldsymbol{p}})x^0 + i\boldsymbol{p}\cdot\boldsymbol{x}}}{p_0 + i\varepsilon} - \frac{e^{-i(p_0-\omega_{\boldsymbol{p}})x^0 - i\boldsymbol{p}\cdot\boldsymbol{x}}}{p_0 - i\varepsilon} \right). \tag{9.17}$$

In the first term we make the substitution $p_0 \to p_0 - \omega_{\boldsymbol{p}}$, and in the second term the substitutions $p_0 \to p_0 + \omega_{\boldsymbol{p}}$ and $\boldsymbol{p} \to -\boldsymbol{p}$:

$$i\Delta_F(x) = i \int \frac{d^4p}{(2\pi)^4 2\omega_{\boldsymbol{p}}} \left(\frac{e^{-ip\cdot x}}{p_0 - (\omega_{\boldsymbol{p}} - i\varepsilon)} - \frac{e^{-ip\cdot x}}{p_0 + \omega_{\boldsymbol{p}} - i\varepsilon} \right) = i \int \frac{d^4p}{(2\pi)^4 2\omega_{\boldsymbol{p}}} \frac{2(\omega_{\boldsymbol{p}} - i\varepsilon)\, e^{-ip\cdot x}}{p_0^2 - (\omega_{\boldsymbol{p}} - i\varepsilon)^2}. \tag{9.18}$$

In the limit $\varepsilon \searrow 0$ we can neglect the ε in the integrand's numerator as this act has no singular effect, approximate $(\omega_{\boldsymbol{p}} - i\varepsilon)^2 \approx \omega_{\boldsymbol{p}}^2 - 2i\omega_{\boldsymbol{p}}\varepsilon$, and write ε instead of $2\omega_{\boldsymbol{p}}\varepsilon$ (this does not change the limit). In total, we find the momentum-space representation of the Feynman propagator of the Klein-Gordon field

$$i\Delta_F(x - y) = i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\varepsilon} \quad (\varepsilon \searrow 0). \tag{9.19}$$

Let us stress that p_0 in this expression is 'off-shell' — it is independent of the three-momentum $\boldsymbol{p}$ ($p_0 \neq \omega_{\boldsymbol{p}}$). The ε in the denominator, although infinitesimally small, must be kept to avoid singularities at $p_0 = \pm\omega_{\boldsymbol{p}}$. It informs us that these poles are shifted from the p_0 axis to locations $\omega_{\boldsymbol{p}} - i\varepsilon$ and $-\omega_{\boldsymbol{p}} + i\varepsilon$ (see Section 9.2.1 below).

From the expression (9.19) it is easy to see that the Feynman propagator is a Green function of the Klein-Gordon equation, since application of the Klein-Gordon differential operator yields

$$(\Box_x + m^2)\Delta_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{-p^2 + m^2}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(x-y)} = -\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} = -\delta(x - y). \tag{9.20}$$

(Note that we could drop the ε since the singularity in the denominator was compensated by the numerator.) This observation can be made also by applying the differential operator $\Box_x + m^2$ to expression (9.12), and carefully differentiating the θ-functions (see Exercise 37).

It is much more practical to work with the momentum space representation of the Feynman propagator, as explicit integration over the momenta leads to rather unpleasant expressions. This is illustrated in Exercise 38 where we treat the case of space-like separated equal-time points x and y. For equal times, using Eq. (9.13), we find that the Feynman propagator is given by the formula

$$i\Delta_F(\boldsymbol{x} - \boldsymbol{y}, 0) = \langle 0|\, \hat{\phi}(\boldsymbol{x}, t)\hat{\phi}(\boldsymbol{y}, t)\, |0\rangle = \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \frac{m}{(2\pi)^2} \frac{K_1(m|\boldsymbol{x} - \boldsymbol{y}|)}{|\boldsymbol{x} - \boldsymbol{y}|}, \tag{9.21}$$

where K_1 is one of the Bessel functions. This function decays exponentially fast for separations $|\boldsymbol{x} - \boldsymbol{y}| \gtrsim \frac{1}{m} = \frac{\hbar}{mc}$, where we have reintroduced $\hbar$ and c to obtain a quantity with dimension length — the (reduced) Compton wavelength of a particle with mass m. In the limit $m \to 0$ one finds a much slower polynomial decay

$$i\Delta_F(\boldsymbol{x} - \boldsymbol{y}, 0)|_{m\to 0} = \frac{1}{(2\pi)^2} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^2}. \tag{9.22}$$

As we have seen, although space-like separated field operators $\hat{\phi}(x)$ commute (they respect microcausality), they still develop nonzero vacuum correlations due to the quantum entanglement of the vacuum state. In this respect let us recall the discussion for a system of coupled quantum oscillators in Section 6.1.3, where it was shown that for the (Gaussian) vacuum state characterized by the matrix A, correlations between different points are given by

$$\langle 0|\,\hat{q}_n\hat{q}_{n'}\,|0\rangle = \frac{\hbar}{2}(\mathsf{A}^{-1})_{nn'}, \quad \text{where} \quad \mathsf{A}^{-1} = \frac{1}{M}\,\mathsf{V}\,\Omega^{-1}\,\mathsf{V}^{\dagger}. \tag{9.23}$$

For a continuous Klein-Gordon field the mode matrix V translates as (see Section 7.1)

$$k \rightsquigarrow \boldsymbol{p} \quad , \quad V_{nk} \rightsquigarrow F(\boldsymbol{x},\boldsymbol{p}) = \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{(2\pi)^{3/2}}, \tag{9.24}$$

and we obtain (with $\hbar = M = 1$)

$$\langle 0|\,\hat{\phi}(\boldsymbol{x})\hat{\phi}(\boldsymbol{y})\,|0\rangle = \frac{1}{2}\int d^3p\, F(\boldsymbol{x},\boldsymbol{p})\frac{1}{\omega_{\boldsymbol{p}}}F^*(\boldsymbol{y},\boldsymbol{p}) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}}e^{-i\boldsymbol{p}\cdot\boldsymbol{y}}, \tag{9.25}$$

in agreement with Eq. (9.21).

Eq. (9.21) shows that the states $\hat{\phi}(\boldsymbol{x},t)\,|0\rangle$, for varying values of $\boldsymbol{x}$, are *not* completely localized (unlike in non-relativistic field theory — see Eq. (9.49)), but rather overlap one another for distances comparable to the Compton wavelength. With this proviso, the Feynman propagator $i\Delta_F(x-y) = \langle 0|\,T\big(\hat{\phi}(x)\hat{\phi}(y)\big)\,|0\rangle$ is a function that specifies the probability amplitude for a particle to travel from a spacetime point y to another spacetime point x (if $x^0 > y^0$), and vice versa (if $y^0 > x^0$).

9.2.1 Retarded propagator

We have mentioned earlier that the $i\varepsilon$ term in Eq. (9.19) serves to avoid the singularities of the denominator when performing the p_0 integration. These singularities are displayed in complex p_0 plane in Fig. 9.1 (left). Let us recall the Cauchy integral theorem,

$$\text{for a closed contour } \Gamma \text{ and a holomorphic function } f: \quad \oint_\Gamma f(z)\,dz = 0, \tag{9.26}$$

which allows us to freely deform the integration contour so long as we do not cross any singularity of the function being integrated. Hence, we may modify the contour of the p_0 integration (the

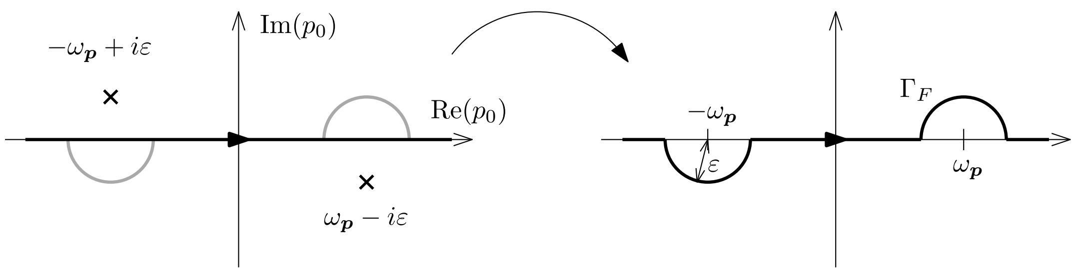


Figure 9.1: Poles of the Feynman propagator avoided by the contour Γ_F.

real axis) so as to avoid the points $\pm\omega_{\boldsymbol{p}}$ on the real line, and obtain the contour Γ_F on the right side of Fig. 9.1. The ε in the denominator can now be sent to zero, and we arrive at the following representation of the Feynman propagator

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\varepsilon} = \int_{\mathbb{R}^3}\frac{d^3p}{(2\pi)^3}\int_{\Gamma_F}\frac{dp_0}{2\pi}\,\frac{e^{-ip\cdot(x-y)}}{p^2-m^2}. \tag{9.27}$$

The Feynman propagator is one of infinitely many Green functions of the Klein-Gordon operator. Any other can be obtained by adding a solution of the homogeneous Klein-Gordon equation. For example, consider the integration contour $\Gamma_R = \Gamma_F + \Gamma_0$ depicted in Fig. 9.2. Since

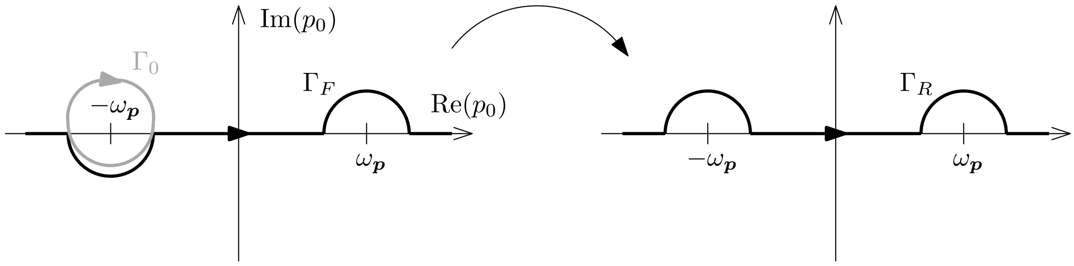


Figure 9.2: Adding the Feynman contour Γ_F and a loop around the point $-\omega_{\boldsymbol{p}}$ (denoted Γ_0) yields the contour Γ_R for the retarded propagator.

$$(\Box_x + m^2)\int_{\mathbb{R}^3}\frac{d^3p}{(2\pi)^3}\oint_{\Gamma_0}\frac{dp_0}{2\pi}\,\frac{e^{-ip\cdot(x-y)}}{p^2-m^2} = \int_{\mathbb{R}^3}\frac{d^3p}{(2\pi)^3}\oint_{\Gamma_0}\frac{dp_0}{2\pi}\,(-1)\,e^{-ip\cdot(x-y)} = 0, \tag{9.28}$$

the function

$$\Delta_R(x-y) = \int_{\mathbb{R}^3}\frac{d^3p}{(2\pi)^3}\int_{\Gamma_R}\frac{dp_0}{2\pi}\,\frac{e^{-ip\cdot(x-y)}}{p^2-m^2} = \int\frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x-y)}}{(p_0+i\varepsilon)^2-\omega_{\boldsymbol{p}}^2} \tag{9.29}$$

is again a Green function of the Klein-Gordon equation, called the *retarder* (or *forward*) *propagator*. An alternative representation is obtained by carrying out the p_0 integration (essentially reversing the steps that led from Eq. (9.14) to Eq. (9.19)):

$$i\Delta_R(x-y) = \theta(x^0-y^0)\int\frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\Big(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\Big)\Big|_{p_0=\omega_{\boldsymbol{p}}} = \theta(x^0-y^0)i\Delta(x-y), \tag{9.30}$$

where $i\Delta(x-y)$ is the Pauli-Jordan commutation function. The retarded propagator is therefore a Green function that propagates a classical field to the future. We shall make use of it in Exercise 39.

9.2.2 Propagators from the action

Rewrite the action of the Klein-Gordon field as a quadratic form

$$S = \int d^4x\left(\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2\right) \mathrel{\hat{=}} \int d^4x\,\frac{1}{2}\phi(x)\big(-\partial_\mu\partial^\mu - m^2\big)\phi(x), \tag{9.31}$$

where we have neglected a boundary term (therefore the sign '$\hat{=}$'). The Feynman propagator Δ_F is a Green's function (an operator inverse) of the bracketed differential operator:

$$(-\Box_x - m^2)\Delta_F(x-y) = \delta(x-y). \tag{9.32}$$

(Note that the minus sign naturally moved from the right-hand side of (9.20) to the left.)

We can formally solve this equation by dividing by the differential operator, adding Feynman's $i\varepsilon$ to avoid singular behaviour, and using the Fourier representation of the δ-function. We obtain

$$\Delta_F(x-y) = \frac{1}{-\Box_x - m^2 + i\varepsilon} \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)} = \int \frac{d^4p}{(2\pi)^4}\, \frac{e^{-ip\cdot(x-y)}}{p_\mu p^\mu - m^2 + i\varepsilon}, \tag{9.33}$$

reproducing formula (9.19).

This way of inferring the Feynman propagator from a free field theory action is rather efficient and indeed very useful, especially when dealing with more complicated field theories (as, for example, in Exercise 40).

9.2.3 Multicomponent Klein-Gordon field

For a multicomponent Klein-Gordon field Φ described by the Lagrangian (7.29), all the components ϕ_r are equivalent, and the creation and annihilation operators corresponding to different components commute. Hence, the Feynman propagator essentially reduces to that of a one-component field:

$$i(\Delta_F)_{rs}(x-y) \equiv \langle 0|\, T\big(\hat\phi_r(x)\hat\phi_s(y)\big)\,|0\rangle = \delta_{rs}\,\langle 0|\, T\big(\hat\phi_1(x)\hat\phi_1(y)\big)\,|0\rangle = \delta_{rs}\, i\Delta_F(x-y). \tag{9.34}$$

A complex Klein-Gordon field is built out of two real (or, after quantization, Hermitian) components according to the formulas (see Section 7.2.1)

$$\begin{pmatrix}\hat\varphi\\ \hat\varphi^\dagger\end{pmatrix} = \mathsf{U}\hat\Phi, \quad \text{where} \quad \hat\Phi = \begin{pmatrix}\hat\phi_1\\ \hat\phi_2\end{pmatrix} \quad \text{and} \quad \mathsf{U} = \frac{1}{\sqrt2}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix} \quad \text{is unitary.} \tag{9.35}$$

Writing Eq. (9.34) in matrix form, and applying U on the left and $\mathsf{U}^\dagger$ on the right, we find

$$\langle 0|\, T\big(\hat\Phi(x)\hat\Phi^\dagger(y)\big)\,|0\rangle = i\Delta_F(x-y)\,\mathbb{I} \quad \to \quad \langle 0|\, T\Big(\mathsf{U}\hat\Phi(x)\big(\mathsf{U}\hat\Phi(y)\big)^\dagger\Big)\,|0\rangle = i\Delta_F(x-y)\,\mathbb{I}. \tag{9.36}$$

Diagonal matrix elements yield

$$\langle 0|\, T\big(\hat\varphi(x)\hat\varphi^\dagger(y)\big)\,|0\rangle = \langle 0|\, T\big(\hat\varphi^\dagger(x)\hat\varphi(y)\big)\,|0\rangle = i\Delta_F(x-y), \tag{9.37}$$

while the off-diagonal ones give

$$\langle 0|\, T\big(\hat\varphi(x)\hat\varphi(y)\big)\,|0\rangle = \langle 0|\, T\big(\hat\varphi^\dagger(x)\hat\varphi^\dagger(y)\big)\,|0\rangle = 0. \tag{9.38}$$

9.3 Feynman propagator of Dirac field

The mode expansions of the Dirac field, Eqs. (8.8) and (8.9), involve two independent types of operators: b and d. The vacuum expectation value of a (time ordered) product of field operators is nonzero only if we take $\hat\Psi$ together with $\hat{\bar\Psi}$. Hence we define the Feynman propagator of the Dirac field

$$i(S_F)_{\alpha\beta}(x-y) = \langle 0|\, T\big(\hat\psi_\alpha(x)\hat{\bar\psi}_\beta(y)\big)\,|0\rangle\,, \tag{9.39}$$

where now the time ordering

$$T\big(\hat\psi_\alpha(x)\hat{\bar\psi}_\beta(y)\big) = \theta(x^0-y^0)\hat\psi_\alpha(x)\hat{\bar\psi}_\beta(y) - \theta(y^0-x^0)\hat{\bar\psi}_\beta(y)\hat\psi_\alpha(x) = \begin{cases} \hat\psi_\alpha(x)\hat{\bar\psi}_\beta(y), & x^0 > y^0\\ -\hat{\bar\psi}_\beta(y)\hat\psi_\alpha(x), & y^0 > x^0\end{cases} \tag{9.40}$$

includes a minus sign, which, similarly as for the normal ordering, ensures that the fermionic fields can be freely anticommuted under the time ordering sign. (Also, with the minus sign S_F will turn out to be a Green's function of the Dirac equation.)

Substituting the mode expansion of the Dirac field we obtain (after a straightforward calculation)

$$i(S_F)_{\alpha\beta}(x-y) = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{p}}}\Big(\theta(x^0-y^0)u_\alpha(\boldsymbol{p},s)\bar{u}_\beta(\boldsymbol{p},s)e^{-ip\cdot(x-y)} - \theta(y^0-x^0)v_\alpha(\boldsymbol{p},s)\bar{v}_\beta(\boldsymbol{p},s)e^{ip\cdot(x-y)}\Big)\Big|_{p_0=\omega_{\boldsymbol{p}}}. \tag{9.41}$$

The sum over the spin s can be carried through by virtue of the identities

$$\sum_s u(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s) = \frac{\not{p}+m}{2m} \quad , \quad \sum_s v(\boldsymbol{p},s)\bar{v}(\boldsymbol{p},s) = \frac{\not{p}-m}{2m}, \tag{9.42}$$

derived in Section 3.2, Eqs. (3.24) and (3.27). (We have used the Feynman 'slash' notation $\gamma^\mu p_\mu \equiv \not{p}$.) This leads to

$$\begin{aligned}
iS_F(x-y) &= \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\Big(\theta(x^0-y^0)(\not{p}+m)e^{-ip\cdot(x-y)} - \theta(y^0-x^0)(\not{p}-m)e^{ip\cdot(x-y)}\Big)\Big|_{p_0=\omega_{\boldsymbol{p}}} \\
&= (i\not{\partial}+m)\int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\Big(\theta(x^0-y^0)e^{-ip\cdot(x-y)} + \theta(y^0-x^0)e^{ip\cdot(x-y)}\Big)\Big|_{p_0=\omega_{\boldsymbol{p}}} \\
&\quad - i\gamma^0\delta(x^0-y^0)\int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\Big(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\Big)\Big|_{p_0=\omega_{\boldsymbol{p}}},
\end{aligned} \tag{9.43}$$

where in the first term we recognize the Feynman propagator of the Klein-Gordon field (see Eqs. (9.14) and (9.19)), and the second term vanishes since the delta function identifies $x^0 = y^0$ and the integrand is then an odd function of $\boldsymbol{p}$:

$$\int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\Big(e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} - e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\Big) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}} 2i\sin\big(\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})\big) = 0. \tag{9.44}$$

Hence,

$$iS_F(x-y) = (i\not{\partial}+m)i\Delta_F(x-y) = i\int \frac{d^4p}{(2\pi)^4}\frac{\not{p}+m}{p^2-m^2+i\varepsilon}\, e^{-ip\cdot(x-y)}. \tag{9.45}$$

This relation between the Dirac and the Klein-Gordon field's Feynman propagators makes it particularly clear that S_F is a Green's function of the Dirac equation, since

$$(i\not{\partial}-m)S_F(x-y) = (i\not{\partial}-m)(i\not{\partial}+m)\Delta_F(x-y) = (-\Box_x - m^2)\Delta_F(x-y) = \delta(x-y). \tag{9.46}$$

(Note that there is also an implicit identity matrix accompanying the δ-function that carries the matrix components of the expression.)

9.4 Feynman propagator of Schrödinger field

For completeness, let us also examine the case of a free non-relativistic (bosonic or fermionic) field $\hat{\psi}(\boldsymbol{x},t)$ introduced in Section 6.2. We note that the function

$$K(\boldsymbol{x},t;\boldsymbol{x}_0,t_0) = \langle 0|\,\hat{\psi}(\boldsymbol{x},t)\hat{\psi}^\dagger(\boldsymbol{x}_0,t_0)\,|0\rangle \tag{9.47}$$

is a solution of the Schrödinger equation,

$$(i\hbar\partial_t - H_{\boldsymbol{x}})K(\boldsymbol{x},t;\boldsymbol{x}_0,t_0) = \langle 0|\,(i\hbar\partial_t - H_{\boldsymbol{x}})\hat{\psi}(\boldsymbol{x},t)\hat{\psi}^\dagger(\boldsymbol{x}_0,t_0)\,|0\rangle = 0 \qquad (H_{\boldsymbol{x}} = -\tfrac{\hbar^2}{2m}\Delta_{\boldsymbol{x}} + V(\boldsymbol{x})) \tag{9.48}$$

with the initial condition

$$K(\boldsymbol{x},t_0;\boldsymbol{x}_0,t_0) = \langle 0|\,\hat{\psi}(\boldsymbol{x},t_0)\hat{\psi}^\dagger(\boldsymbol{x}_0,t_0)\,|0\rangle = \langle 0|\,[\hat{\psi}(\boldsymbol{x},t_0),\hat{\psi}^\dagger(\boldsymbol{x}_0,t_0)]_\mp\,|0\rangle = \delta(\boldsymbol{x}-\boldsymbol{x}_0). \tag{9.49}$$

(Here $[\ ,\ ]_- \equiv [\ ,\ ]$ corresponds to bosons, while $[\ ,\ ]_+ \equiv \{\ ,\ \}$ to fermions, and a use has been made of Eqs. (6.31) and (6.52), respectively.) In the second step we used the fact that the non-relativistic field operator consists of annihilation operators only, $\hat{\psi}(\boldsymbol{x},t) = \sum_k \hat{a}_k(t)u_k(\boldsymbol{x})$, and so it annihilates the vacuum state, $\hat{\psi}(\boldsymbol{x},t)\,|0\rangle = 0$. In the last step we employed the bosonic commutation (6.28), or fermionic anticommutation relations (6.51), respectively. Note that unlike in the relativistic case, Eq. (9.21), the overlap between the states $\hat{\psi}^\dagger(\boldsymbol{x},t)\,|0\rangle$ with different $\boldsymbol{x}$ is zero (the states are completely local).

The non-relativistic Feynman propagator G_F is defined as usual,

$$iG_F(\boldsymbol{x},t;\boldsymbol{x}_0,t_0) = \langle 0|\,T\big(\hat{\psi}(\boldsymbol{x},t)\hat{\psi}^\dagger(\boldsymbol{x}_0,t_0)\big)\,|0\rangle = \theta(t-t_0)K(\boldsymbol{x},t;\boldsymbol{x}_0,t_0), \tag{9.50}$$

where the term with $\theta(t_0-t)$ has automatically vanished. A simple differentiation shows that this is a Green's function of the Schrödinger equation:

$$(i\hbar\partial_t - H_{\boldsymbol{x}})iG_F(\boldsymbol{x},t;\boldsymbol{x}_0,t_0) = i\hbar\delta(t-t_0)\delta(\boldsymbol{x}-\boldsymbol{x}_0). \tag{9.51}$$

It is worth to note that the quantity K is commonly referred to as the quantum-mechanical propagator, and it can be represented by the quantum-mechanical Feynman path integral

$$K(\boldsymbol{x}_b,t_b;\boldsymbol{x}_a,t_a) = \int_{\boldsymbol{x}(t_a)=\boldsymbol{x}_a}^{\boldsymbol{x}(t_b)=\boldsymbol{x}_b} \mathcal{D}\boldsymbol{x}(t)\, e^{\frac{i}{\hbar}S[\boldsymbol{x}(t)]}, \quad \text{where} \quad S[\boldsymbol{x}(t)] = \int_{t_a}^{t_b} dt\,\Big(\frac{m}{2}\dot{\boldsymbol{x}}^2(t) - V(\boldsymbol{x}(t))\Big) \tag{9.52}$$

is the classical action. It gives the probability amplitude of transition from a point $\boldsymbol{x}_a$ at time t_a to a point $\boldsymbol{x}_b$ at time t_b.

9.5 Exercises

Exercise 36. *Integral representation of the Heaviside step function.* Verify the following integral representation of the Heaviside step function:

$$\theta(t) = \lim_{\varepsilon\to 0_+} \frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-itx}}{x+i\varepsilon}\,dx = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases} . \tag{9.53}$$

Solution:

We provide two derivations as both of them are conceptually interesting.

1. Application of the Sokhotski formula (with infinitesimal $\varepsilon > 0$) ,

$$\frac{1}{x+i\varepsilon} = P\frac{1}{x} - i\pi\delta(x), \tag{9.54}$$

yields

$$\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-itx}}{x+i\varepsilon} = \frac{i}{2\pi}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}\frac{e^{-itx}}{x}\,dx+\frac{1}{2} = \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}\frac{\sin(tx)}{2\pi x}\,dx+\frac{1}{2} = \int_0^{\infty}\frac{\sin(tx)}{\pi x}\,dx+\frac{1}{2}. \tag{9.55}$$

The remaining integral, known as the Dirichlet integral, is given by (formula 3.721:1 in [12])

$$\int_0^{\infty}\frac{\sin(tx)}{x}\,dx = \operatorname{sgn}(t)\int_0^{\infty}\frac{\sin(x)}{x}\,dx = \operatorname{sgn}(t)\frac{\pi}{2}. \tag{9.56}$$

Altogether we have

$$\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-itx}}{x+i\varepsilon}\,dx = \frac{\operatorname{sgn}(t)}{2}+\frac{1}{2} = \theta(t) \tag{9.57}$$

(with $\theta(0)=\frac{1}{2}$).

2. Alternatively, we make use of the methods of complex contour integration, namely the Cauchy integral formulas for a holomorphic function $f(z)$,

$$\oint_\Gamma f(z)\,dz = 0 \quad\text{and}\quad \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-z_0}\,dz = f(z_0) \quad (z_0 \text{ counterclockwise encircled by } \Gamma). \tag{9.58}$$

In our problem, $f(z)=e^{-itz}$ and $z_0=-i\varepsilon$, where $\varepsilon>0$.

For $t>0$ we take a sequence of (counterclockwise oriented) semicircles of radius R, Γ_R, that close in the lower half-plane, and hence enclose the pole. We have $z=x$, $dz=dx$ along the segment from R to $-R$, and $z=Re^{i\varphi}$, $dz=izd\varphi$ along the arc. Thus,

$$e^{-it(-i\varepsilon)} = \frac{1}{2\pi i}\oint_{\Gamma_R}\frac{e^{-itz}}{z+i\varepsilon}\,dz = \frac{1}{2\pi i}\int_{+R}^{-R}\frac{e^{-itx}}{x+i\varepsilon}\,dx + \int_{\pi}^{2\pi}\frac{iRe^{i\varphi}}{2\pi i}\frac{e^{-itR\cos\varphi}e^{tR\sin\varphi}}{R\,e^{i\varphi}+i\varepsilon}\,d\varphi. \tag{9.59}$$

Now, crucially, the second integral on the right-hand side vanishes in limit $R\to\infty$ (due to $t\sin\varphi$ being negative for $\varphi\in(\pi,2\pi)$, and the integrand being dominated in absolute value by a constant, e.g., $\frac{1}{\pi}$). So, we conclude that

$$e^{-t\varepsilon} = \frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-itx}}{x+i\varepsilon}\,dx \quad (t>0). \tag{9.60}$$

For $t < 0$ we take (counterclockwise oriented) semicircles that close in the upper half-plane, so that $t \sin\varphi$ is again negative. These contours do not encircle the pole, and so we have

$$0 = \int_{-R}^{+R} \frac{e^{-itx}}{x + i\varepsilon} dx + \int_0^{\pi} \frac{iRe^{i\varphi} e^{-itR\cos\varphi} e^{tR\sin\varphi}}{R\, e^{i\varphi} + i\varepsilon} d\varphi \quad \xrightarrow{R\to\infty} \quad \int_{-\infty}^{+\infty} \frac{e^{-itx}}{x + i\varepsilon} dx = 0 \quad (t < 0). \tag{9.61}$$

Combining the two cases and taking the limit $\varepsilon \to 0_+$ yields Eq. (9.53).

Exercise 37. *Feynman propagator is a Green's function.* Show (by a direct calculation) that the Feynman propagator of the Klein-Gordon field

$$i\Delta_F(x - y) = \langle 0| T\big(\hat\phi(x)\hat\phi(y)\big) |0\rangle \tag{9.62}$$

satisfies the equation

$$(\Box_x + m^2)\Delta_F(x - y) = -\delta(x - y). \tag{9.63}$$

Solution:

We take into account the definition of time ordering,

$$T\big(\hat\phi(x)\hat\phi(y)\big) = \theta(x^0 - y^0)\hat\phi(x)\hat\phi(y) + \theta(y^0 - x^0)\hat\phi(y)\hat\phi(x), \tag{9.64}$$

and the rule for differentiation of the Heaviside step function (all derivatives in this exercise will be with respect to the x coordinates), $\partial_0\theta(x^0 - y^0) = \delta(x^0 - y^0)$, to find

$$\begin{aligned}
\partial_0 T\big(\hat\phi(x)\hat\phi(y)\big) &= \delta(x^0 - y^0)\hat\phi(x)\hat\phi(y) - \delta(y^0 - x^0)\hat\phi(y)\hat\phi(x) \\
&\quad + \theta(x^0 - y^0)\partial_0\hat\phi(x)\hat\phi(y) + \theta(y^0 - x^0)\hat\phi(y)\partial_0\hat\phi(x) \\
&= \delta(x^0 - y^0)\ \big[\hat\phi(x), \hat\phi(y)\big]\Big|_{x^0 = y^0} + T\big(\partial_0\hat\phi(x)\hat\phi(y)\big) \\
&= T\big(\partial_0\hat\phi(x)\hat\phi(y)\big),
\end{aligned} \tag{9.65}$$

where we have used the equal-time commutation relations (9.1). Similarly,

$$\begin{aligned}
\partial_0^2 T\big(\hat\phi(x)\hat\phi(y)\big) &= \delta(x^0 - y^0)\ \big[\partial_0\hat\phi(x), \hat\phi(y)\big]\Big|_{x^0 = y^0} + T\big(\partial_0^2\hat\phi(x)\hat\phi(y)\big) \\
&= -i\,\delta(x^0 - y^0)\delta(\boldsymbol{x} - \boldsymbol{y}) + T\big(\partial_0^2\hat\phi(x)\hat\phi(y)\big),
\end{aligned} \tag{9.66}$$

so finally we observe that

$$\begin{aligned}
(\Box_x + m^2)\Delta_F(x - y) &= (\partial_0^2 - \partial_i\partial_i + m^2)(-i)\,\langle 0| T\big(\hat\phi(x)\hat\phi(y)\big) |0\rangle \\
&= (-i)^2\delta(x - y)\langle 0|0\rangle - i\,\langle 0| T\big((\partial_0^2 - \partial_i\partial_i + m^2)\hat\phi(x)\hat\phi(y)\big) |0\rangle \\
&= -\delta(x - y).
\end{aligned} \tag{9.67}$$

Exercise 38. *Feynman propagator of the Klein-Gordon field in position space.* Show that for equal-time space-like separated points the Feynman propagator of the Klein-Gordon field is given by the formula

$$i\Delta_F(\boldsymbol{x} - \boldsymbol{y}, 0) = \frac{m}{(2\pi)^2}\frac{K_1(m|\boldsymbol{x} - \boldsymbol{y}|)}{|\boldsymbol{x} - \boldsymbol{y}|}, \tag{9.68}$$

where K_1 is a Bessel function. Find the limit $m \to 0$.

Solution:

According to Eq. (9.21) we need to calculate the integral

$$i\Delta_F(\boldsymbol{x}-\boldsymbol{y},0) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\, e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \int_0^\infty \frac{dP}{(2\pi)^3}\frac{P^2}{2\sqrt{P^2+m^2}}\int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta\, e^{iPR\cos\theta}, \tag{9.69}$$

where we have denoted $R \equiv |\boldsymbol{x}-\boldsymbol{y}|$, and introduces spherical coordinates ($P \equiv |\boldsymbol{p}|, \theta, \varphi$) in the momentum space with the 'p_z-axis' pointing along the vector $\boldsymbol{x}-\boldsymbol{y}$. The angular integrations can be carried out yielding

$$i\Delta_F(\boldsymbol{x}-\boldsymbol{y},0) = \int_0^\infty \frac{dP}{(2\pi)^2}\frac{P^2}{2\sqrt{P^2+m^2}}\left[-\frac{e^{iPR\cos\theta}}{iPR}\right]_0^\pi = \frac{1}{R}\int_0^\infty \frac{dP}{(2\pi)^2}\frac{P\sin(PR)}{\sqrt{P^2+m^2}}. \tag{9.70}$$

Further rearrangements reveal an integral representation of the Bessel function K_0 (see formula 3.754:2 in Ref. [12]):

$$i\Delta_F(\boldsymbol{x}-\boldsymbol{y},0) = -\frac{1}{(2\pi)^2}\frac{1}{R}\frac{\partial}{\partial R}\int_0^\infty dP\,\frac{\cos(PR)}{\sqrt{P^2+m^2}} = -\frac{1}{(2\pi)^2}\frac{1}{R}\frac{\partial}{\partial R}K_0(mR) = \frac{m}{(2\pi)^2}\frac{K_1(mR)}{R}, \tag{9.71}$$

where in the last step we used formula 8.486:18 of [12].

The asymptotics of the K_1 function are given by formulas 8.446 and 8.451:6 of [12],

$$K_1(z) \sim \frac{1}{z} \quad (\text{for } z \ll 1) \qquad \text{and} \qquad K_1(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z} \quad (\text{for } z \gg 1). \tag{9.72}$$

In the limit $m \to 0$ we therefore obtain

$$\lim_{m\to 0} i\Delta_F(\boldsymbol{x}-\boldsymbol{y},0) = \frac{1}{(2\pi)^2}\frac{1}{R^2}. \tag{9.73}$$

Exercise 39. *Particle creation by a classical source.* Consider the Lagrangian

$$\mathscr{L}(\phi, \partial_\mu\phi, x) = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + J(x)\phi \tag{9.74}$$

of a one-component Klein-Gordon field coupled to an external (Schwinger) source $J(x)$, which is an explicit classical function of x. Assume that $J(x)$ vanishes outside of the time interval $(0, T)$. Let $|0\rangle$ denote the vacuum state before $t = 0$, i.e., before the intervention of the source J.

For $t < 0$ the vacuum expectation value of the total four-momentum clearly vanishes, $\langle 0|\,\hat{P}_\mu(t)\,|0\rangle = 0$. Determine

$$\langle 0|\,\hat{P}_\mu(t)\,|0\rangle \quad \text{for} \quad t > T. \tag{9.75}$$

(Note that $\hat{P}_\mu(t)$, and its expectation values, are not a priori conserved quantities since the Lagrangian (9.74) depends explicitly on time.)

Solution:

The equation of motion for the given Lagrangian reads

$$(\Box_x + m^2)\hat{\phi}(x) = J(x). \tag{9.76}$$

(The source J can therefore be interpreted as an external driving force.) A general solution consists of the solution of the homogeneous problem $\hat{\phi}_0(x)$, given by the mode expansion (9.2), and a particular solution given by the convolution of the right-hand side with a Green's function of the Klein-Gordon operator. We use the retarded propagator Δ_R, which satisfies $(\Box_x + m^2)\Delta_R(x - y) = -\delta(x-y)$, and write

$$\hat{\phi}(x) = \hat{\phi}_0(x) - \int d^4y\, \Delta_R(x-y) J(y). \tag{9.77}$$

Eqs. (9.2) and (9.30) then yield

$$\begin{aligned}\hat{\phi}(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}} \Big(\hat{a}_{\boldsymbol{p}}\, e^{-ip\cdot x} + \hat{a}^\dagger_{\boldsymbol{p}}\, e^{ip\cdot x} \Big) \\ &\quad + \int d^4y\, J(y)\, i\theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^3 2\omega_{\boldsymbol{p}}} \Big(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \Big).\end{aligned} \tag{9.78}$$

Here, and also in the following, $p_0 = \omega_{\boldsymbol{p}}$.

We are interested in times $x^0 = t > T$, for which the source J "has passed". In this case $\theta(x^0 - y^0)$ is effectively equal to 1, and we can write

$$\hat{\phi}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}} \Big(\hat{a}'_{\boldsymbol{p}} e^{-ip\cdot x} + \hat{a}'^\dagger_{\boldsymbol{p}} e^{ip\cdot x} \Big), \quad \hat{a}'_{\boldsymbol{p}} = \hat{a}_{\boldsymbol{p}} + \frac{i\,\tilde{J}(p)}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}}, \quad \tilde{J}(p) = \int d^4y\, J(y) e^{ip\cdot y}. \tag{9.79}$$

Now, since the commutation relations for $\hat{a}'_{\boldsymbol{p}}$, $\hat{a}'^\dagger_{\boldsymbol{p}}$ and $\hat{a}_{\boldsymbol{p}}$, $\hat{a}^\dagger_{\boldsymbol{p}}$ are identical, we can immediately write the formula for the total four-momentum operator at times $t > T$ (cf. Eq. (7.22)):

$$\hat{P}_\mu(t) = \int d^3p\, p_\mu\, \hat{a}'^\dagger_{\boldsymbol{p}} \hat{a}'_{\boldsymbol{p}} = \int d^3p\, p_\mu \left(\hat{a}^\dagger_{\boldsymbol{p}} - \frac{i\,\tilde{J}^*(p)}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}} \right) \left(\hat{a}_{\boldsymbol{p}} + \frac{i\,\tilde{J}(p)}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}} \right). \tag{9.80}$$

The ensuing vacuum expectation value reads

$$\langle 0 |\, \hat{P}_\mu(t)\, | 0 \rangle = \int \frac{d^3p\, p_\mu}{(2\pi)^3 2\omega_{\boldsymbol{p}}}\, |\tilde{J}(p)|^2 \qquad (t > T), \tag{9.81}$$

which is nonzero. The classical source has excited the quantum field, i.e., has created particles. With respect to the annihilation operators $\hat{a}'_{\boldsymbol{p}}$ the state $|0\rangle$ is no longer 'empty': $\hat{a}'_{\boldsymbol{p}}\, |0\rangle \neq 0$.

Chapter 10

Canonical quantization of electromagnetic field

10.1 Classical electromagnetism

The 'Maxwell' Lagrangian

$$\mathscr{L}_M(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu} \tag{10.1}$$

yields the equations of motion

$$\frac{\partial \mathscr{L}_M}{\partial A_\nu} - \partial_\mu \frac{\partial \mathscr{L}_M}{\partial(\partial_\mu A_\nu)} = \frac{1}{2}\partial_\mu \left(\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} F^{\rho\sigma} \right) = \frac{1}{2}\partial_\mu \Big((\delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma) F^{\rho\sigma} \Big) = \partial_\mu F^{\mu\nu} = 0, \tag{10.2}$$

which is a half of the (sourceless) Maxwell equations of electrodynamics. The other half is a consequence of the definition of the Faraday tensor $F_{\mu\nu}$ in terms of the electromagnetic four-potential A_μ, and of commutativity of partial derivatives:

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0. \tag{10.3}$$

Traditional form of the Maxwell equations can be recovered by relating $F_{\mu\nu}$ to the electric and magnetic field intensities $\boldsymbol{E}$ and $\boldsymbol{B}$ via Eq. (4.2).

In terms of A_μ the Maxwell equations (10.2) read

$$\partial_\mu F^\mu{}_\nu = \Box A_\nu - \partial_\mu \partial_\nu A^\mu = (g_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\mu = 0, \tag{10.4}$$

which has a particularly simple form with the Lorenz gauge condition imposed,

$$\partial_\mu A^\mu = 0 \quad \rightarrow \quad \Box A^\nu = 0. \tag{10.5}$$

(This gauge condition is named after Ludvig Lorenz, not to be confused with Hendrik Lorentz of Lorentz transformations. Clearly, the Lorenz condition is Lorentz-invariant.) However, without a gauge fixing condition, the differential operator $g_{\mu\nu}\Box - \partial_\mu\partial_\nu$ does not have an inverse (i.e., a Green's function), and so the Feynman propagator does not exist. This is because any four-potential of the form $A_\mu = \partial_\mu \Lambda$, where $\Lambda(x)$ is an arbitrary function lies in the kernel of this operator. The problem stems from gauge invariance of electromagnetism, $A_\mu \to A_\mu + \partial_\mu \Lambda$.

One way to resolve this issue is to fix a gauge (commonly, the Coulomb gauge $A_0 = 0$, $\partial_i A^i = 0$), and quantize only the remaining degrees of freedom (recall the advanced quantum mechanics course). This approach, however, lacks manifest Lorentz covariance, and is therefore not suitable for perturbative treatment of interacting relativistic quantum field theories.

To retain manifest Lorentz covariance we aim to use the Lorenz gauge condition $\partial_\mu A^\mu = 0$, but, rather than imposing this gauge explicitly, we define a modified Lagrangian with an extra 'gauge fixing' term:

$$\mathscr{L}_\xi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\rho A^\rho)^2, \quad \text{where} \quad \xi \in \mathbb{R} \tag{10.6}$$

is, for the moment, an arbitrary parameter. The corresponding equations of motion read

$$\frac{\partial \mathscr{L}_\xi}{\partial A_\nu} - \partial_\mu \frac{\partial \mathscr{L}_\xi}{\partial(\partial_\mu A_\nu)} = \partial_\mu F^{\mu\nu} + \frac{1}{\xi}\,\partial_\mu \left((\partial_\rho A^\rho) \frac{\partial(\partial_\sigma A_\kappa)}{\partial(\partial_\mu A_\nu)} g^{\sigma\kappa} \right) = \partial_\mu F^{\mu\nu} + \frac{1}{\xi}\,\partial^\nu(\partial_\rho A^\rho) = 0, \tag{10.7}$$

or, in terms of the four-potential,

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{\xi}\,\partial^\nu(\partial_\mu A^\mu) = \Box A^\nu - \left(1 - \frac{1}{\xi}\right)\partial^\nu(\partial_\mu A^\mu) = 0. \tag{10.8}$$

This reduces to a simple equation

$$\Box A^\nu = 0 \tag{10.9}$$

for the choice of the parameter $\xi = 1$, referred to as the *Feynman gauge*.

We will assume $\xi = 1$ from now on, and hence work with the Lagrangian

$$\begin{aligned}
\mathscr{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\rho A^\rho)^2 \\
&= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) - \frac{1}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu) \\
&= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}\partial_\mu(A_\nu \partial^\nu A^\mu - A^\mu \partial_\nu A^\nu),
\end{aligned} \tag{10.10}$$

which is equivalent, after dropping the inessential four-divergence term, to a collection of four real massless Klein-Gordon fields:

$$\mathscr{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) = -\frac{1}{2}(\partial_\mu A_0)(\partial^\mu A_0) + \frac{1}{2}(\partial_\mu A_i)(\partial^\mu A_i). \tag{10.11}$$

It is important, however, to mind the minus sign of the zeroth component. Also, note that A_μ is not a multiplet of scalar fields, but a vector field that transforms in the spin-1 representation of the Lorentz group,

$$A'^\mu(x') = L^\mu{}_\nu A^\nu(x). \tag{10.12}$$

The canonical momenta and the Hamiltonian corresponding to the Lagrangian (10.11) are

$$\pi^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_0 A_\mu)} = -\partial_0 A^\mu \quad , \quad H = -\frac{1}{2}\int d^3x \left((\pi^0)^2 + (\nabla A_0)^2 \right) + \frac{1}{2}\sum_{i=1}^{3} \int d^3x \left((\pi^i)^2 + (\nabla A_i)^2 \right). \tag{10.13}$$

The zeroth component of the field is therefore a source of negative energy.

It should be emphasized that the theory with Lagrangian (10.6) only becomes equivalent with Maxwell's electromagnetism (Lagrangian (10.1)) once the Lorenz condition $\partial_\mu A^\mu = 0$ is imposed. Then, the equations of motion (10.9) coincide with Eq. (10.5). Imposing this condition in the realm of quantum theory requires some care, as we shall see below in Section 10.2.

It is worth to mention that the methods of path-integral quantization provide a very elegant (and in fact superior) way of handling gauge invariant theories; not only electromagnetism, but also more general non-Abelian gauge theories. These methods will be explained in the advanced quantum field theory course (KTPA2).

10.2 Covariant canonical quantization

We impose the usual equal-time commutation relations for the quantized canonical fields $\hat{A}_\mu$ and $\hat{\pi}^\mu = -\partial_0 \hat{A}^\mu$:

$$[\hat{A}_\mu(\boldsymbol{x},t), \hat{\pi}^\nu(\boldsymbol{y},t)] = i\,\delta_\mu^\nu\,\delta(\boldsymbol{x}-\boldsymbol{y}) \quad , \quad [\hat{A}_\mu(\boldsymbol{x},t), \hat{A}_\nu(\boldsymbol{y},t)] = [\hat{\pi}^\mu(\boldsymbol{x},t), \hat{\pi}^\nu(\boldsymbol{y},t)] = 0. \tag{10.14}$$

A general solution of the equation of motion (10.9) has the mode expansion

$$\hat{A}_\mu(x) = \sum_{\lambda=0}^{3} \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{k}}}} \Big(\hat{a}_{\boldsymbol{k},\lambda}\, \varepsilon_\mu(\boldsymbol{k},\lambda)\, e^{-ik\cdot x} + \hat{a}^\dagger_{\boldsymbol{k},\lambda}\, \varepsilon_\mu(\boldsymbol{k},\lambda)\, e^{ik\cdot x} \Big), \quad \text{where} \quad k_0 = \omega_{\boldsymbol{k}} = |\boldsymbol{k}|, \tag{10.15}$$

and where the polarisation vectors $\varepsilon^\mu(\boldsymbol{k},\lambda)$, assumed real for simplicity, form (for each value of $\boldsymbol{k}$) a tetrad indexed by $\lambda = 0,1,2,3$, which is

$$\text{orthonormal:} \quad \varepsilon^\mu(\boldsymbol{k},\lambda)\, \varepsilon_\mu(\boldsymbol{k},\lambda') = g_{\lambda\lambda'}, \tag{10.16}$$

$$\text{and complete:} \quad \sum_{\lambda,\lambda'=0}^{3} \varepsilon_\mu(\boldsymbol{k},\lambda)\, g_{\lambda\lambda'}\, \varepsilon_\nu(\boldsymbol{k},\lambda') = g_{\mu\nu}. \tag{10.17}$$

The time-like unit vector corresponds to $\lambda = 0$, and the space-like unit vectors to $\lambda = i = 1,2,3$. Specifically, we may choose

$$\varepsilon^\mu(\boldsymbol{k},0) = (1,\boldsymbol{0}) \quad \text{and} \quad \varepsilon^\mu(\boldsymbol{k},i) = (0,\boldsymbol{\varepsilon}_i(\boldsymbol{k})), \qquad \text{where} \qquad \boldsymbol{\varepsilon}_3(\boldsymbol{k}) = \frac{\boldsymbol{k}}{|\boldsymbol{k}|} \quad , \quad \boldsymbol{\varepsilon}_i(\boldsymbol{k})\cdot\boldsymbol{\varepsilon}_j(\boldsymbol{k}) = \delta_{ij}. \tag{10.18}$$

Commutation relations between $\hat{a}_{\boldsymbol{k},\lambda}$ and $\hat{a}^\dagger_{\boldsymbol{k},\lambda}$, compatible with the canonical relations (10.14), are inferred as follows. First, let us cast the mode expansion (10.15) as

$$\hat{A}_\mu(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{k}}}} \Big(\hat{\mathsf{a}}_{\mu,\boldsymbol{k}}\, e^{-ik\cdot x} + \hat{\mathsf{a}}^\dagger_{\mu,\boldsymbol{k}}\, e^{ik\cdot x} \Big), \quad \text{where} \quad \hat{\mathsf{a}}_{\mu,\boldsymbol{k}} \equiv \sum_{\lambda=0}^{3} \hat{a}_{\boldsymbol{k},\lambda}\, \varepsilon_\mu(\boldsymbol{k},\lambda) \tag{10.19}$$

are the annihilation operators corresponding to the individual field components (labelled by μ). Remembering the case of a multicomponent Klein-Gordon field, Eq. (7.33), we need to ascertain that these creation and annihilation operators obey the relation

$$[\hat{\mathsf{a}}_{\mu,\boldsymbol{k}}, \hat{\mathsf{a}}^\dagger_{\nu,\boldsymbol{k}'}] = \sum_{\lambda,\lambda'=0}^{3} \varepsilon_\mu(\boldsymbol{k},\lambda)\, \varepsilon_\nu(\boldsymbol{k}',\lambda')\, [\hat{a}_{\boldsymbol{k},\lambda}, \hat{a}^\dagger_{\boldsymbol{k}',\lambda'}] = -g_{\mu\nu}\, \delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{10.20}$$

Therefore, taking into account Eq. (10.17), the creation and annihilation operators corresponding to the individual polarization states (labelled by λ) must satisfy

$$[\hat{a}_{\boldsymbol{k},\lambda}, \hat{a}^\dagger_{\boldsymbol{k}',\lambda'}] = -g_{\lambda\lambda'}\, \delta(\boldsymbol{k}-\boldsymbol{k}') \quad , \quad [\hat{a}_{\boldsymbol{k},\lambda}, \hat{a}_{\boldsymbol{k}',\lambda'}] = 0 \quad , \quad [\hat{a}^\dagger_{\boldsymbol{k},\lambda}, \hat{a}^\dagger_{\boldsymbol{k}',\lambda'}] = 0. \tag{10.21}$$

Since the Lagrangian (10.11) is a sum of independent massless Klein-Gordon Lagrangians (where the zeroth component carries a negative sign), the total four-momentum operator is simply given by a sum of four-momentum operators (7.22) (corresponding to a single-component Klein-Gordon field), with additional minus sign for the zeroth component:

$$\hat{P}_\mu = \int d^3k\, k_\mu (-g_{\nu\rho}) \mathsf{\hat{a}}^\dagger_{\nu,\boldsymbol{k}} \mathsf{\hat{a}}_{\rho,\boldsymbol{k}} = \int d^3k\, k_\mu \left(-\hat{a}^\dagger_{\boldsymbol{k},0}\hat{a}_{\boldsymbol{k},0} + \sum_{\lambda=1}^{3} \hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda} \right), \quad \text{where} \quad k_0 = \omega_{\boldsymbol{k}}. \tag{10.22}$$

(Here we have used the definition of $\mathsf{\hat{a}}_{\mu,\boldsymbol{k}}$ in Eq. (10.19), and the orthonormality relation (10.16).) In particular, $\hat{P}_0 = :\hat{H}:$ is the normal-ordered Hamiltonian. Our current theory has 4 (independent) polarization states, one of which ($\lambda = 0$) carries negative energies. The electromagnetic field, on the contrary, has only 2 polarizations (typically linear or circular), which are positive in energy.

In order to obtain a quantum theory of electromagnetic field we need to impose the Lorenz gauge condition. However, it is not hard to realize that the commutation relations (10.14) are incompatible with $\partial^\mu \hat{A}_\mu(x) = 0$ imposed on the operator level, since

$$[\partial^\mu \hat{A}_\mu(\boldsymbol{x},t), \hat{A}_\nu(\boldsymbol{y},t)] = [-\hat{\pi}^0(\boldsymbol{x},t), \hat{A}_\nu(\boldsymbol{y},t)] + [\partial^i \hat{A}_i(\boldsymbol{x},t), \hat{A}_\nu(\boldsymbol{y},t)] = i\,\delta^0_\nu\,\delta(\boldsymbol{x}-\boldsymbol{y}) \neq 0. \tag{10.23}$$

The correct procedure is more subtle. First, we make the decomposition $\hat{A}_\mu(x) = \hat{A}^{(+)}_\mu(x) + \hat{A}^{(-)}_\mu(x)$, where

$$\hat{A}^{(+)}_\mu(x) = \sum_{\lambda=0}^{3} \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{k}}}}\, \hat{a}_{\boldsymbol{k},\lambda}\, \varepsilon_\mu(\boldsymbol{k},\lambda)\, e^{-ik\cdot x} \quad , \quad \hat{A}^{(-)}_\mu(x) = \left(\hat{A}^{(+)}_\mu(x)\right)^\dagger, \tag{10.24}$$

i.e., $\hat{A}^{(+)}_\mu$ contains all the annihilation operators, whereas $\hat{A}^{(-)}_\mu$ all the creation operators. Next, we impose the so-called Gupta–Bleuler condition, which has to hold true for all ‘physical’ states $|\alpha\rangle$ in the Fock space:

$$\partial^\mu \hat{A}^{(+)}_\mu(x)\,|\alpha\rangle = 0, \quad \text{which implies} \quad \langle\alpha|\,\partial^\mu \hat{A}_\mu(x)\,|\alpha\rangle = \langle\alpha|\,\partial^\mu \hat{A}^{(-)}_\mu(x) + \partial^\mu \hat{A}^{(+)}_\mu(x)\,|\alpha\rangle = 0. \tag{10.25}$$

That is, the Lorenz condition holds on the level of expectation values of physical states.

The Gupta-Bleuler condition written out in detail implies

$$\sum_{\lambda=0}^{3} \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{k}}}}\, \hat{a}_{\boldsymbol{k},\lambda}\,(-ik^\mu)\,\varepsilon_\mu(\boldsymbol{k},\lambda)\,e^{-ik\cdot x}\,|\alpha\rangle = 0 \quad \rightarrow \quad \sum_{\lambda=0}^{3} k^\mu \varepsilon_\mu(\boldsymbol{k},\lambda)\,\hat{a}_{\boldsymbol{k},\lambda}\,|\alpha\rangle = 0. \tag{10.26}$$

This gets simplified further, since by Eq. (10.18)

$$k^\mu\varepsilon_\mu(\boldsymbol{k},1) = k^\mu\varepsilon_\mu(\boldsymbol{k},2) = 0 \quad , \quad k^\mu\varepsilon_\mu(\boldsymbol{k},3) = -|\boldsymbol{k}| \quad , \quad k^\mu\varepsilon_\mu(\boldsymbol{k},0) = k^0 = |\boldsymbol{k}|. \tag{10.27}$$

We obtain

$$(\hat{a}_{\boldsymbol{k},0} - \hat{a}_{\boldsymbol{k},3})\,|\alpha\rangle = 0 \quad \rightarrow \quad \langle\alpha|\,\hat{a}^\dagger_{\boldsymbol{k},0}\hat{a}_{\boldsymbol{k},0}\,|\alpha\rangle = \langle\alpha|\,\hat{a}^\dagger_{\boldsymbol{k},3}\hat{a}_{\boldsymbol{k},3}\,|\alpha\rangle\,, \tag{10.28}$$

which reduces the total four-momentum of physical states to a sum of only two (transverse) polarizations,

$$\langle\alpha|\,\hat{P}_\mu\,|\alpha\rangle = \int d^3k\, k_\mu \sum_{\lambda=1}^{2} \langle\alpha|\,\hat{a}^\dagger_{\boldsymbol{k},\lambda}\hat{a}_{\boldsymbol{k},\lambda}\,|\alpha\rangle\,. \tag{10.29}$$

This gives energy that is always positive.

The Feynman propagator of the electromagnetic field $\hat{A}_\mu$ is for the Feynman "gauge" $\xi = 1$ simply related (due to its similarity with a multicomponent Klein-Gordon field, Section 9.2.3) to the Feynman propagator of a one-component Klein-Gordon field with mass $m = 0$, Eq. (9.19). With regard to the commutation relation of Eq. (10.20) we find

$$i(D_F)_{\mu\nu}(x-y) = \langle 0|\, T\big(\hat{A}_\mu(x)\hat{A}_\nu(y)\big)\,|0\rangle = -g_{\mu\nu}\, i\Delta_F(x-y)|_{m=0} = i\int \frac{d^4k}{(2\pi)^4}\frac{-g_{\mu\nu}}{k^2+i\varepsilon}\, e^{-ik\cdot(x-y)}. \tag{10.30}$$

The same result is derived in Exercise 40 directly from the Lagrangian (for generic ξ), rewriting Eq. (10.6) in the form

$$\begin{aligned}\mathscr{L}_\xi &= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) - \frac{1}{2\xi}(\partial_\mu A^\mu)(\partial_\nu A^\nu) \\ &\hat{=} \frac{1}{2}A^\mu\left[g_{\mu\nu}\Box - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]A^\nu,\end{aligned} \tag{10.31}$$

and inverting the bracketed differential operator. Note that the Gupta-Bleuler condition plays no role in the derivation of the Feynman propagator.

10.3 Proca's massive vector field

A massive spin-1 field is characterized by the Proca Lagrangian

$$\mathscr{L}_P = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu, \quad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{10.32}$$

and where the additional mass term breaks gauge invariance of the theory. The corresponding Euler-Lagrange equations read

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = (\Box + m^2)A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0. \tag{10.33}$$

Acting further with ∂_ν implies (for $m \neq 0$)

$$\partial_\nu A^\nu = 0 \qquad \text{and hence} \qquad (\Box + m^2)A^\nu = 0. \tag{10.34}$$

That is, the Lorentz condition $\partial_\mu A^\mu = 0$ is a *consequence* of the equations of motion, which in turn reduce to a multicomponent Klein-Gordon equation (with extra condition $\partial_\mu A^\mu = 0$).

Plane-wave solutions of the form $A_\mu(x) = \varepsilon_\mu(\boldsymbol{k},\lambda)\, e^{\pm ik\cdot x}$ satisfy

$$k^0 = \omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + m^2} \qquad \text{and} \qquad k^\mu \varepsilon_\mu(\boldsymbol{k},\lambda) = 0, \tag{10.35}$$

which leads to three independent polarization states

$$\varepsilon^\mu(\boldsymbol{k},1) = (0, \boldsymbol{\varepsilon}_1(\boldsymbol{k})) \quad , \quad \varepsilon^\mu(\boldsymbol{k},2) = (0, \boldsymbol{\varepsilon}_2(\boldsymbol{k})) \quad , \quad \varepsilon^\mu(\boldsymbol{k},3) = \Big(\frac{|\boldsymbol{k}|}{m}, \frac{\omega_{\boldsymbol{k}}}{m}\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\Big), \tag{10.36}$$

where $\boldsymbol{k}\cdot\boldsymbol{\varepsilon}_{1,2}(\boldsymbol{k}) = 0$. The polarizations $\varepsilon^\mu(\boldsymbol{k},\lambda)$ are (with respect to the direction of motion $\boldsymbol{k}$) transverse for $\lambda = 1, 2$, and longitudinal for $\lambda = 3$. In the 'massless' limit $m \to 0$, the Proca Lagrangian $\mathscr{L}_P$ reduces to the Maxwell Lagrangian $\mathscr{L}_M$ of Eq. (10.1). For the longitudinal polarization, and the corresponding mode, we have

$$m\,\varepsilon^\mu(\boldsymbol{k},3)|_{m\to 0} = (|\boldsymbol{k}|, \boldsymbol{k}) \quad \to \quad A_\mu(x) = k_\mu\, e^{\pm ik\cdot x} = \mp i\partial_\mu e^{\pm ik\cdot x}. \tag{10.37}$$

But the latter is merely a 'pure gauge' — it yields zero electromagnetic field, $F_{\mu\nu} = 0$. That is, only the two transverse polarizations survive the massless limit.

For canonical quantization of the Proca field see Ref. [2, Ch. 6]. Here we only mention the result for the Feynman propagator, which follows directly from the Lagrangian (10.32), casting

$$\begin{aligned}\mathscr{L}_P &= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) + \frac{1}{2}m^2 A_\mu A^\mu \\ &\hat{=} \frac{1}{2}A^\mu\Big(g_{\mu\nu}(\Box + m^2) - \partial_\mu\partial_\nu\Big)A^\nu. \end{aligned} \tag{10.38}$$

In Exercise 40 we find

$$D^{\mu\nu}_{F,P}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, \tilde{D}^{\mu\nu}_{F,P}(k)\, e^{-ik\cdot(x-y)}, \quad \text{with} \quad \tilde{D}^{\mu\nu}_{F,P}(k) \,\hat{=}\, \frac{-g^{\nu\rho} + \frac{k^\nu k^\rho}{m^2}}{k^2 - m^2 + i\varepsilon}. \tag{10.39}$$

This can be useful even when dealing with massless vector bosons (photons), if the term $\frac{k^\nu k^\rho}{m^2}$ disappears during calculations.

10.4 Standard model particles

The free fields that we have studied so far describe all elementary particles of the Standard model:

1. Spin-0 Klein-Gordon field: Higgs boson.
2. Spin-$\frac{1}{2}$ Dirac field: 6 quarks (up, down, charm, strange, top, bottom), 6 leptons (electron, muon, tau, and respective neutrinos), and their antiparticles.
3. Spin-1 massless field: photon, 8 gluons.
4. Spin-1 massive Proca field: gauge bosons $W^\pm$ and Z .

The quarks are composed to yield a "zoo" of other subatomic particles — the hadrons. These fall in two categories, mesons (made of two quarks) and baryons (made of three quarks, for example neutron and proton).

10.5 Exercises

Exercise 40. *Photon propagator for arbitrary parameter ξ and mass m.* Find a Green function of the operator

$$g_{\mu\nu}(\Box + m^2) - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu \tag{10.40}$$

(with the Feynman's $i\varepsilon$ prescription), and study the limits $\xi \to \infty$ and $m \to 0$.

Solution:

We look for a solution of the equation

$$\left[g_{\mu\nu}(\Box + m^2) - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu\right] D_F^{\nu\rho}(x-y) = \delta_\mu^\rho\,\delta(x-y). \tag{10.41}$$

In the Fourier representation

$$D_F^{\nu\rho}(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\tilde{D}_F^{\nu\rho}(k)\,e^{-ik\cdot(x-y)} \quad , \quad \delta(x-y) = \int \frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}, \tag{10.42}$$

we have an algebraic equation

$$\left[-\delta_\nu^\mu(k^2 - m^2) + \left(1 - \frac{1}{\xi}\right)k^\mu k_\nu\right](\tilde{D}_F)^\nu{}_\rho(k) = \delta_\rho^\mu. \tag{10.43}$$

We are looking for an inverse of a matrix of the form $a\mathbb{I} + b\mathsf{K}$, where $(\mathsf{K})^\mu{}_\nu = \frac{k^\mu k_\nu}{k^2}$ is a projector, $\mathsf{K}^2 = \mathsf{K}$. In this respect it is useful to realise that any function of a projector is of the same form,

$$f(x) = \sum_{n=0}^{\infty} f_n x^n \quad \to \quad f(\mathsf{K}) = f_0\mathbb{I} + \Big(\sum_{n=1}^{\infty} f_n\Big)\mathsf{K}, \tag{10.44}$$

and hence make the ansatz

$$(\tilde{D}_F)^\nu{}_\rho(k) = \alpha\,\delta_\rho^\nu + \beta\,k^\nu k_\rho. \tag{10.45}$$

Plugging this into Eq. (10.43) yields

$$-(k^2 - m^2)\alpha\,\delta_\rho^\mu - (k^2 - m^2)\beta\,k^\mu k_\rho + \left(1 - \frac{1}{\xi}\right)\alpha\,k^\mu k_\rho + \left(1 - \frac{1}{\xi}\right)\beta k^2\,k^\mu k_\rho = \delta_\rho^\mu, \tag{10.46}$$

which fixes the unknown coefficients as

$$\alpha = -\frac{1}{k^2 - m^2} \quad , \quad \beta = \frac{1-\xi}{k^2 - m^2}\,\frac{1}{k^2 - \xi m^2}. \tag{10.47}$$

With Feynman's $i\varepsilon$ prescription in the numerators, we finally arrive at the Fourier-space Feynman propagator

$$\tilde{D}_F^{\nu\rho}(k) = -\frac{g^{\nu\rho}}{k^2 + i\varepsilon - m^2} + \frac{(1-\xi)k^\nu k^\rho}{(k^2 + i\varepsilon - m^2)(k^2 + i\varepsilon - \xi m^2)}. \tag{10.48}$$

Let us study several limiting cases:

1. $\xi \to \infty$ yields the Feynman propagator of Proca's massive vector field

$$\tilde{D}_{F,P}^{\nu\rho}(k) = \frac{-g^{\nu\rho} + \frac{k^\nu k^\rho}{m^2}}{k^2 - m^2 + i\varepsilon}. \tag{10.49}$$

2. $m \to 0$ yields the Feynman propagator for massless photon, and arbitrary ξ

$$\tilde{D}^{\nu\rho}_{F,\xi}(k) = -\frac{g^{\nu\rho}}{k^2 + i\varepsilon} + (1-\xi)\frac{k^\nu k^\rho}{(k^2 + i\varepsilon)^2}. \tag{10.50}$$

   In particular, for

   (a) $\xi = 1$ we recover the propagator in Feynman 'gauge', Eq. (10.30) ,

   (b) $\xi = 0$ we obtain the propagator in so-called Landau 'gauge',

   $$\tilde{D}^{\nu\rho}_{F,\xi=0}(k) = \frac{-g^{\nu\rho}k^2 + k^\nu k^\rho}{(k^2 + i\varepsilon)^2}, \tag{10.51}$$

   which is transverse in four dimensions: $k_\nu \tilde{D}^{\nu\rho}_{F,\xi=0}(k) = 0$,

   (c) $\xi \to \infty$ (the Maxwell Lagrangian (10.1)) we hit a singularity — the differential operator (10.40) cannot be inverted. This fact is visible in Eq. (10.43), where the matrix $-g_{\mu\nu}k^2 + k_\mu k_\nu$ has nonzero kernel, namely, when applied to k^ν it gives 0.

Chapter 11

Interacting quantum fields

The previous chapters were dedicated to (canonical) quantization of ‘free’ fields, whose Lagrangians were at most quadratic in the fields and their derivatives. The corresponding equations of motion, since linear, were relatively easy to solve in terms of normal modes (plane waves), and these modes (whose quantum excitations were interpreted as particles) evolved independently of one another. So far, quantum field theory has been a relatively simple subject.

To describe interparticle interactions we need to include higher-order terms that constitute an interaction part of the Lagrangian. As a simple example, let us consider the Lagrangian

$$\mathscr{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2}_{\mathscr{L}_0} \underbrace{- \frac{\lambda}{4!}\phi^4}_{\mathscr{L}_I}, \tag{11.1}$$

which describes a self-interacting real scalar field ϕ. (The non-relativistic limit, of an analogous theory with complex field $\varphi(x)$, gives a field theory that describes an ensemble of non-relativistic particles interacting via contact interaction, Eq. (6.69)). The ensuing equation of motion

$$(\Box + m^2)\phi + \frac{\lambda}{3!}\phi^3 = 0 \tag{11.2}$$

is nonlinear, and although certain exact solutions can be obtained, they do not allow to evolve arbitrary initial conditions.

A general strategy for dealing with interacting field theories is to divide the Lagrangian into a free (quadratic) part $\mathscr{L}_0$, and an interacting part $\mathscr{L}_I$ (see, for example, Eq. (11.1)), and treat the interacting part as a perturbation. In quantum theory we correspondingly divide the full Hamiltonian,

$$\hat{H} = \hat{H}_0 + \hat{H}_I, \tag{11.3}$$

set up the interaction (Dirac) picture, and develop the perturbation (Dyson) series. An overview of this standard procedure is provided in the following section.

In the case of Lagrangian (11.1), $\hat{H}_0$ is the total Hamiltonian of a free Klein-Gordon field (7.2), and

$$\hat{H}_I(t) = \int d^3x\, \frac{\lambda}{4!}\hat{\phi}^4(\boldsymbol{x}, t). \tag{11.4}$$

More generally, if $\mathscr{L}_I$ does not contain time derivatives of fields then (according to Eqs. (5.39) and (5.40))

$$H = \int d^3x \left(\frac{\partial \mathscr{L}_0}{\partial(\partial_0 \phi_r)}\, \partial_0 \phi_r - \mathscr{L}_0 - \mathscr{L}_I \right) = H_0 + H_I, \quad \text{where} \quad H_I = -\int d^3x\, \mathscr{L}_I. \tag{11.5}$$

11.1 Interaction picture

We assume, for simplicity, that the Hamiltonian H has no explicit time dependence.

Consider a state $|\alpha\rangle \equiv |\alpha(0)\rangle$ and an operator $\hat{A} \equiv \hat{A}(0)$ at time $t = 0$. Their time evolution depends on which of the three basic pictures of quantum theory (indicated by a superscript) is chosen:

1. Schrödinger picture

$$\begin{aligned} |\alpha(t)\rangle^S &= e^{-it\hat{H}} |\alpha\rangle , \\ \hat{A}^S(t) &= \hat{A}, \end{aligned} \tag{11.6}$$

where the state evolves under the full Hamiltonian $\hat{H}$, while the operator stays constant;

2. Heisenberg picture

$$\begin{aligned} |\alpha(t)\rangle^H &= |\alpha\rangle , \\ \hat{A}^H(t) &= e^{it\hat{H}} \hat{A}\, e^{-it\hat{H}}, \end{aligned} \tag{11.7}$$

where the state remains constant, and the operator evolves under the full Hamiltonian $\hat{H}$;

3. Interaction (or Dirac) picture

$$\begin{aligned} |\alpha(t)\rangle^I &= e^{it\hat{H}_0^S} |\alpha(t)\rangle^S = e^{it\hat{H}_0^S} e^{-it\hat{H}} |\alpha\rangle , \\ \hat{A}^I(t) &= e^{it\hat{H}_0^S} \hat{A}\, e^{-it\hat{H}_0^S} = e^{it\hat{H}_0^S} e^{-it\hat{H}} \hat{A}^H(t)\, e^{it\hat{H}} e^{-it\hat{H}_0^S}, \end{aligned} \tag{11.8}$$

where the operator evolves as in the Heisenberg picture, but only under the free Hamiltonian $\hat{H}_0^S$. (Note that since $\hat{H}_0$ in general does not commute with the total Hamiltonian $\hat{H} \equiv \hat{H}^S = \hat{H}^H$, it undergoes nontrivial time evolution $\hat{H}_0^H(t)$ in the Heisenberg picture.)

The above definitions relating states and operators in the three pictures are based on the requirement that expectation values must be the same regardless of the picture chosen:

$${}^S\langle\alpha(t)|\, \hat{A}^S(t)\, |\alpha(t)\rangle^S = {}^H\langle\alpha(t)|\, \hat{A}^H(t)\, |\alpha(t)\rangle^H = {}^I\langle\alpha(t)|\, \hat{A}^I(t)\, |\alpha(t)\rangle^I . \tag{11.9}$$

The three pictures agree at (arbitrarily chosen) time $t = 0$.

In the interaction picture, operators, such as, for concreteness, the scalar field operator $\hat{\phi}^I(\boldsymbol{x}, t)$, evolve according to the equation

$$\partial_t \hat{\phi}^I = -i[\hat{\phi}^I, \hat{H}_0^S] = -i[\hat{\phi}^I, \hat{H}_0^I], \tag{11.10}$$

where we used the fact that $\hat{H}_0^I(t) = e^{it\hat{H}_0^S} \hat{H}_0^S\, e^{-it\hat{H}_0^S} = \hat{H}_0^S$. This equation has the form of a Heisenberg equation of motion governed solely by the free part of the Hamiltonian. The field $\hat{\phi}^I$ is therefore given by the same mode expansion as in the free-field case, Eq. (7.14), in contrast to the Heisenberg-picture field $\hat{\phi}^H(\boldsymbol{x}, t)$, whose evolution is driven by the full Hamiltonian $\hat{H}$. The 'full' Heisenberg equation is nonlinear, and in practise intractable (recall Eq. (6.63) for a system of interacting non-relativistic particles).

The evolution of states in the interaction picture is easily inferred from the Schrödinger picture via relation (11.8):

$$|\alpha(t)\rangle^S = e^{-i(t-t_0)\hat{H}} |\alpha(t_0)\rangle^S \quad \to \quad |\alpha(t)\rangle^I = e^{it\hat{H}_0^S} e^{-i(t-t_0)\hat{H}} e^{-it_0\hat{H}_0^S} |\alpha(t_0)\rangle^I . \tag{11.11}$$

Here
$$\hat{U}(t,t_0) \equiv e^{it\hat{H}_0^S} e^{-i(t-t_0)\hat{H}} e^{-it_0\hat{H}_0^S} \tag{11.12}$$
is the evolution operator of the interaction picture, also known as the *Dyson operator* (not to be confused with $e^{-i(t-t_0)\hat{H}}$, which is the evolution operator of the Schrödinger picture). The operator (or rather family of operators) $\hat{U}(t,t_0)$ is clearly unitary (it is a composition of unitary operators), and satisfies the relations
$$\hat{U}(t_0,t_0) = 1 \quad , \quad \hat{U}(t_2,t_1)\hat{U}(t_1,t_0) = \hat{U}(t_2,t_0) \quad , \quad \hat{U}^{-1}(t_1,t_0) = \hat{U}(t_0,t_1). \tag{11.13}$$
It is a unique solution of the differential equation
$$i\partial_t \hat{U}(t,t_0) = \hat{H}_I^I(t)\hat{U}(t,t_0) \quad , \quad \hat{U}(t_0,t_0) = 1, \tag{11.14}$$
which results from the following calculation:
$$\partial_t \hat{U}(t,t_0) = e^{it\hat{H}_0^S}(i\hat{H}_0^S - i\hat{H})e^{-i(t-t_0)\hat{H}} e^{-it_0\hat{H}_0^S} = -i\, e^{it\hat{H}_0^S}\hat{H}_I^S e^{-it\hat{H}_0^S}\hat{U}(t,t_0) = -i\,\hat{H}_I^I(t)\hat{U}(t,t_0). \tag{11.15}$$

The solution of Eq. (11.14) can be cast in three different ways, producing three different representations of the Dyson operator. First of them, Eq. (11.12), features the exponential of the full Hamiltonian $\hat{H}$, which in practice is intractable. Second is based on the infinitesimal relation
$$\hat{U}(t+\varepsilon,t_0) \approx \hat{U}(t,t_0) - i\varepsilon\hat{H}_I^I(t)\hat{U}(t,t_0) \approx e^{-i\varepsilon\hat{H}_I^I(t)}\hat{U}(t,t_0), \tag{11.16}$$
which, when used 'infinitely many times', evolves $\hat{U}(t_0,t_0) = 1$ to
$$\hat{U}(t,t_0) = \lim_{N\to\infty} e^{-i\varepsilon\hat{H}_I^I(t_{N-1})}\dots e^{-i\varepsilon\hat{H}_I^I(t_1)}e^{-i\varepsilon\hat{H}_I^I(t_0)}, \quad \text{where} \quad \varepsilon \equiv \frac{t-t_0}{N} = t_{n+1}-t_n, \quad t_N \equiv t. \tag{11.17}$$
It should be stressed that $\hat{H}_I^I(t)$ taken at different times in general do not commute, and so the product of exponentials cannot be reduced to $\exp\big(-i\int_{t_0}^t dt\,\hat{H}_I^I(t)\big)$. The representation (11.17) is often useful in theoretical considerations.

The third representation of $\hat{U}(t,t_0)$ is obtained by formal integration of Eq. (11.14), and repeated substitutions of the right-hand side:
$$\begin{aligned}\hat{U}(t,t_0) &= 1 + (-i)\int_{t_0}^t dt_1 \hat{H}_I^I(t_1)\hat{U}(t_1,t_0)\\ &= 1 + (-i)\int_{t_0}^t dt_1\hat{H}_I^I(t_1) + (-i)^2\int_{t_0}^t dt_1\int_{t_0}^{t_1} dt_2 \hat{H}_I^I(t_1)\hat{H}_I^I(t_2)\hat{U}(t_2,t_0)\\ &\;\vdots\\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^t dt_1\dots\int_{t_0}^t dt_n\, T\big(\hat{H}_I^I(t_1)\dots\hat{H}_I^I(t_n)\big) \equiv T\exp\left(-i\int_{t_0}^t dt'\,\hat{H}_I^I(t')\right),\end{aligned} \tag{11.18}$$
where we employed a general (bosonic) time ordering formula
$$T\big(\hat{A}_1(t_1)\dots\hat{A}_n(t_n)\big) = \hat{A}_{\sigma(1)}(t_{\sigma(1)})\dots\hat{A}_{\sigma(n)}(t_{\sigma(n)}), \quad \text{where} \quad \sigma\in\mathcal{S}_n: \quad t_{\sigma(1)}\geq\dots\geq t_{\sigma(n)}, \tag{11.19}$$
to rewrite the nested time integrations as
$$\underbrace{\int_{t_0}^t dt_1\int_{t_0}^{t_1} dt_2\dots\int_{t_0}^{t_{n-1}} dt_n}_{\int\limits_{t>t_1>\dots>t_n>t_0} dt_1\dots dt_n}\,\hat{H}_I^I(t_1)\hat{H}_I^I(t_2)\dots\hat{H}_I^I(t_n) = \frac{1}{n!}\int_{t_0}^t dt_1\dots\int_{t_0}^t dt_n\, T\big(\hat{H}_I^I(t_1)\dots\hat{H}_I^I(t_n)\big). \tag{11.20}$$

The representation (11.18), known as the *Dyson series*, is well suited for perturbative calculations. It is an expansion in the (presumably small) perturbation $\hat{H}^I_I$. For the ϕ^4 interaction Hamiltonian (11.4) the size of the perturbation is controlled by the parameter λ (the 'coupling constant', or 'interaction strength').

11.1.1 Scattering matrix

For asymptotic times $t_0 \to -\infty$, $t \to +\infty$ we introduce the $\hat{S}$ operator and the S matrix (*scattering matrix*):

$$\hat{S} = \hat{U}(+\infty, -\infty) \qquad \text{and} \qquad S_{fi} = \langle \alpha_f | \, \hat{S} \, | \alpha_i \rangle \, . \tag{11.21}$$

S_{fi} is the overlap between an initial state $|\alpha_i\rangle$ after it has been evolved by $\hat{S}$, and some desired final state $|\alpha_f\rangle$, i.e., the probability amplitude (the 'scattering amplitude') of the process under consideration. Typically, the states describe a group of particles with well-defined on-shell four-momenta:

$$|\alpha\rangle = |\boldsymbol{p}_1, \ldots \boldsymbol{p}_n\rangle \propto \hat{a}^\dagger_{\boldsymbol{p}_1} \ldots \hat{a}^\dagger_{\boldsymbol{p}_n} |0\rangle \, , \tag{11.22}$$

where we have considered only scalar particles for simplicity, and omitted normalization factors (see Eq. (7.60)).

In Chapter 12 we will evaluate a few example scattering amplitudes S_{fi} in the lowest (non-trivial) order of the Dyson perturbation expansion (11.18). In general, the creation operators in Eq. (11.22) can be recast in terms of the fields operators $\hat{\phi}^I$, and so the scattering amplitudes, at certain perturbative order, get expressed in terms of vacuum expectation values of time-ordered products of fields taken at various spacetime points,

$$\langle 0 | \, T\big(\hat{\phi}^I(x_1) \ldots \hat{\phi}^I(x_n)\big) \, |0\rangle \, . \tag{11.23}$$

An exact formula and its derivation (which goes under the name *LSZ reduction*) will be provided in the advanced quantum field theory course (02KTP2), together with the so-called *Gell-Mann–Low formula* that takes into account differences between the free-theory vacuum $|0\rangle$, and the vacuum of the full interacting theory (which we neglect for now).

11.2 Wick theorem

The Wick theorem provides a way how to express time-ordered products of field operators $\hat{\phi}^I$ (which, according to Eq. (11.10), are effectively free fields) in terms of normal-ordered products and Feynman propagators. This is a crucial simplification in calculations of the scattering amplitudes S_{fi} (but also other physical predictions) within the perturbation theory.

All field operators will be by default taken in the interaction picture, and so we shall omit the superscript "I". For simplicity, we will initially focus on a one-component real Klein-Gordon field $\hat{\phi} \equiv \hat{\phi}^I$. The Wick theorem, in its 'generating' form proven in Exercise 41, states that

$$T\exp\Big(i\int d^4x J(x)\hat{\phi}(x)\Big) = :\exp\Big(i\int d^4x J(x)\hat{\phi}(x)\Big): \; \exp\Big(-\frac{1}{2}\int d^4x d^4y J(x)\, i\Delta_F(x-y)\, J(y)\Big), \tag{11.24}$$

where J is an arbitrary real-valued function of a spacetime point (i.e., a generic Schwinger source — see Exercise 39). From this formula, time-ordered products $T\big(\hat{\phi}(x_1) \ldots \hat{\phi}(x_n)\big)$ can be obtained by taking variational derivatives,

$$\begin{aligned}\frac{\delta}{i\delta J(y)} T\exp\Big(i\int d^4x\,J(x)\hat{\phi}(x)\Big) &= \sum_{n=1}^{\infty}\frac{i^n}{n!}\int d^4x_1\dots d^4x_n\,\frac{\delta\big(J(x_1)\dots J(x_n)\big)}{i\delta J(y)}\,T\big(\hat{\phi}(x_1)\dots\hat{\phi}(x_n)\big)\\ &= \sum_{n=1}^{\infty}\frac{i^{n-1}}{n!}n\int d^4x_1\dots d^4x_{n-1}\,J(x_1)\dots J(x_{n-1})\,T\big(\hat{\phi}(x_1)\dots\hat{\phi}(x_{n-1})\hat{\phi}(y)\big)\\ &= T\Big[\hat{\phi}(y)\exp\Big(i\int d^4x\,J(x)\hat{\phi}(x)\Big)\Big], \end{aligned} \tag{11.25}$$

successively at n spacetime points, and setting $J(x)=0$ $(\forall x)$.

Equivalently, one can expand both sides of Eq. (11.24), and identify the (symmetrized) coefficients of a monomial $J(x_1)\dots J(x_n)$. For example, in second order in J we have

$$\begin{aligned}\frac{i^2}{2!}\int d^4x_1d^4x_2\,J(x_1)J(x_2)\,T\big(\hat{\phi}(x_1)\hat{\phi}(x_2)\big) = \frac{i^2}{2!}&\int d^4x_1d^4x_2\,J(x_1)J(x_2)\ :\hat{\phi}(x_1)\hat{\phi}(x_2):\\ -\frac{1}{2}&\int d^4x_1d^4x_2\,J(x_1)J(x_2)\,i\Delta_F(x_1-x_2), \end{aligned} \tag{11.26}$$

which gives

$$T\big(\hat{\phi}(x_1)\hat{\phi}(x_2)\big) = :\hat{\phi}(x_1)\hat{\phi}(x_2):\ + \langle 0|\,T\big(\hat{\phi}(x_1)\hat{\phi}(x_2)\big)\,|0\rangle\,. \tag{11.27}$$

Symmetrization is trivial in this case as bosonic fields freely commute under both time and normal ordering, and the Feynman propagator of the Klein-Gordon field, $i\Delta_F(x_1-x_2) = \langle 0|\,T\big(\hat{\phi}(x_1)\hat{\phi}(x_2)\big)\,|0\rangle$, is symmetric: $\Delta_F(x_1-x_2)=\Delta_F(x_2-x_1)$ (recall the representation (9.19), and substitute $p\to -p$). In Exercise 42, similar calculation provides a formula for a four-point product of fields.

To state the result for an arbitrary number of points $x_1,\dots,x_n$ let us introduce the shorthand notation

$$\begin{aligned} T(1\dots n) &\equiv T\big(\hat{\phi}(x_1)\dots\hat{\phi}(x_n)\big)\\ :1\dots n: &\equiv :\hat{\phi}(x_1)\dots\hat{\phi}(x_n):\\ \langle 1\dots n\rangle &\equiv \langle 0|\,T\big(\hat{\phi}(x_1)\dots\hat{\phi}(x_n)\big)\,|0\rangle\,. \end{aligned} \tag{11.28}$$

Generalizing the formulas (11.27) and (11.68) it turns out that: for n even,

$$\begin{aligned} T(1\dots n) = &:1\dots n:\\ &+\langle 12\rangle\ :3\dots n:\ + \text{(1-fold contractions)}\\ &\ \vdots\\ &+\langle 12\rangle\dots\langle n{-}1\ n\rangle + (\tfrac{n}{2}\text{-fold contractions}), \end{aligned} \tag{11.29}$$

while for n odd,

$$\begin{aligned} T(1\dots n) = &:1\dots n:\\ &+\langle 12\rangle\ :3\dots n:\ + \text{(1-fold contractions)}\\ &\ \vdots\\ &+\langle 12\rangle\dots\langle n{-}2\ n{-}1\rangle\ :n:\ + (\tfrac{n-1}{2}\text{-fold contractions}). \end{aligned} \tag{11.30}$$

These formulas look cumbersome, but in practise are easy to implement. A k-fold contraction is a way of selecting k pairs out of the numbers $1, \ldots, n$ to form propagators; leaving the remaining numbers under the symbol of normal ordering. We start with no contractions (all fields under normal ordering), and in each line increase the number of contracted pairs (i.e., the number of propagators), while summing over all possibilities ('contraction patterns'). The combinatorics is nice enough not to produce any numerical factors in front of the individual terms.

11.2.1 Vacuum expectation values

Taking vacuum expectation value of the Wick theorem, Eq. (11.24), renders the normal-ordered exponential equal to 1, and so we obtain the formula

$$Z_0[J] \equiv \langle 0| T\exp\Big(i\int d^4x\, J(x)\hat\phi(x)\Big)|0\rangle = \exp\Big(-\frac{1}{2}\int d^4x\, d^4y\, J(x)\, i\Delta_F(x-y)\, J(y)\Big) \quad (11.31)$$

for the so-called *generating functional* $Z_0[J]$ of a free quantum field theory. This identity, together with Eq. (11.25), can be used to express the *n-point (correlation) functions*

$$\begin{aligned}\langle 0| T\big(\hat\phi(x_1)\ldots\hat\phi(x_n)\big)|0\rangle &= \frac{\delta}{i\delta J(x_1)}\cdots\frac{\delta}{i\delta J(x_n)}\bigg|_{J=0}\langle 0|T\exp\Big(i\int d^4x\, J(x)\hat\phi(x)\Big)|0\rangle \\ &= \frac{\delta}{i\delta J(x_1)}\cdots\frac{\delta}{i\delta J(x_n)}\bigg|_{J=0}\exp\Big(\frac{1}{2}\int d^4x\, d^4y\, iJ(x)\, i\Delta_F(x-y)\, iJ(y)\Big)\end{aligned} \quad (11.32)$$

in terms of sums of products of propagators (i.e., 2-point functions $\langle 0|T\big(\hat\phi(x)\hat\phi(y)\big)|0\rangle$).

Let us calculate the right-hand side explicitly by expanding the exponential, and keeping only the terms of n-th order in J. All odd-point functions clearly vanish (in agreement with Eq. (11.30)) as there are no odd powers of J,

$$\langle 0|T\big(\hat\phi(x_1)\ldots\hat\phi(x_n)\big)|0\rangle = 0 \quad (\forall n \text{ odd}), \quad (11.33)$$

and for n even we obtain

$$\begin{aligned}\langle 0|T\big(\hat\phi(x_1)\ldots\hat\phi(x_n)\big)|0\rangle &= \frac{1}{2^{n/2}(n/2)!}\int d^4y_1\ldots d^4y_n\,\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}\big(J(y_1)\ldots J(y_n)\big) \\ &\quad\times i\Delta_F(y_1-y_2)\ldots i\Delta_F(y_{n-1}-y_n) \\ &= \frac{1}{2^{n/2}(n/2)!}\int d^4y_1\ldots d^4y_n\sum_{\sigma\in\mathcal{S}_n}\delta(y_1-x_{\sigma(1)})\ldots\delta(y_n-x_{\sigma(n)}) \\ &\quad\times i\Delta_F(y_1-y_2)\ldots i\Delta_F(y_{n-1}-y_n) \\ &= \frac{1}{2^{n/2}(n/2)!}\sum_{\sigma\in\mathcal{S}_n} i\Delta_F(x_{\sigma(1)}-x_{\sigma(2)})\ldots i\Delta_F(x_{\sigma(n-1)}-x_{\sigma(n)}).\end{aligned} \quad (11.34)$$

The resulting sum over all permutations of n elements decomposes into subsets of $2^{n/2}(n/2)!$ identical summands, since the order of propagators is irrelevant, as well as the order of points within each propagator Δ_F (due to its symmetricity). Hence we can reduce the sum to only those permutations that give different $\frac{n}{2}$-fold contraction patterns $\sigma \in \mathcal{C}_n$:

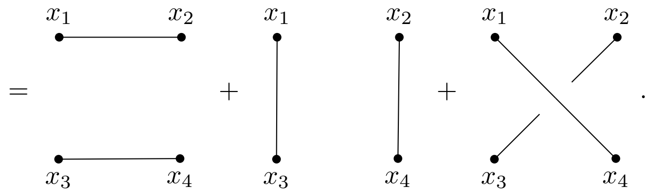
$$\langle 0|T\big(\hat\phi(x_1)\ldots\hat\phi(x_n)\big)|0\rangle = \sum_{\sigma\in\mathcal{C}_n} i\Delta_F(x_{\sigma(1)}-x_{\sigma(2)})\ldots i\Delta_F(x_{\sigma(n-1)}-x_{\sigma(n)}) \quad (\forall n \text{ even}). \quad (11.35)$$

This result agrees with the last line in Eq. (11.29) (which is the only term on the right-hand side that survives taking vacuum expectation value).

It is convenient (and also common) to represent the various terms of the sum in Eq. (11.35) graphically. For each contraction pattern we draw a diagram with n points numbered $x_1, \ldots, x_n$, and $n/2$ edges representing the individual contractions. With the rule that an edge between points i and j carries a factor $i\Delta_F(x_i - x_j)$, an n-point correlation function of a real scalar field can be expressed diagrammatically as a sum over all labelled unoriented graphs with n vertices and $n/2$ edges, where each vertex has degree (i.e., the number of neighbours) equal to 1.

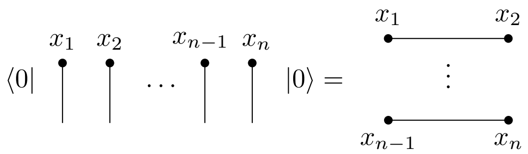
As an illustrative example, the 4-point correlation function can be expressed, with the short-hand notation of Eq. (11.28), as

$$\langle 1234 \rangle = \langle 12 \rangle\langle 34 \rangle + \langle 13 \rangle\langle 24 \rangle + \langle 14 \rangle\langle 23 \rangle$$



(11.36)

The general result (11.35) can then be depicted as



$+ \quad (\tfrac{n}{2}\text{-fold contractions}).$ (11.37)

Finally, it is worth to mention that the Wick theorem (11.31) provides a passage between the canonical (operator) formulation of quantum field theory, and the path-integral formulation (which will be studied extensively in the advanced course 02KTPA2). To see the main point, recall Eqs. (6.79) and (6.82) from Exercise 23, which, with $\hbar = 1$ and the replacements $\mathsf{A} \to -\frac{i}{2}\mathsf{A}$, $\vec{j} \to i\,\vec{j}$, produce the formula

$$\mathcal{N}^2 \int d^N q \, \exp\left(\frac{i}{2}\vec{q}^{\,T}\mathsf{A}\,\vec{q} + i\,\vec{j}^{\,T}\vec{q}\right) = \exp\left(-\frac{i}{2}\vec{j}^{\,T}\mathsf{A}^{-1}\vec{j}\right). \tag{11.38}$$

Its continuum version with

$$\vec{q} \rightsquigarrow \phi(x) \quad , \quad \mathsf{A} \rightsquigarrow (-\Box_x - m^2)\delta(x-y) \quad , \quad \mathsf{A}^{-1} \rightsquigarrow \Delta_F(x-y) \tag{11.39}$$

(recall Eqs. (9.31) and (9.32)) reads (absorbing the normalization factor $\mathcal{N}^2$ in the definition of the measure $\mathcal{D}\phi(x)$)

$$\begin{aligned} \int \mathcal{D}\phi(x)\, e^{i\int d^4x\, \frac{1}{2}\phi(-\Box_x - m^2)\phi + i\int d^4x\, J(x)\phi(x)} &= \exp\Big(-\frac{1}{2}\int d^4x\, d^4y\, J(x)\, i\Delta_F(x-y)\, J(y)\Big) \\ &= \langle 0|\, T\exp\Big(i\int d^4x\, J(x)\hat{\phi}(x)\Big)\,|0\rangle\,. \end{aligned} \tag{11.40}$$

The left-hand side — the *field-theoretic Feynman path integral* (or *functional integral*) — is an ∞-fold integral over the values of a classical field ϕ at all spacetime points x. The exponent consists of the free-theory action plus the 'source term' (see the Lagrangian (9.74) in Exercise 39).

11.2.2 Multiple fields

The Wick theorem extends seamlessly to fields with arbitrary many components (see Eq. (11.64) in Exercise 41), and also to fermionic fields using Grassmann-valued sources [3, Ch. 4-2]. General time ordering formula for fermionic operators $\hat{B}_i(t_i)$ reads

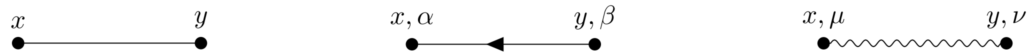
$$T\big(\hat{B}_1(t_1)\dots\hat{B}_n(t_n)\big) = \mathrm{sgn}(\sigma)\hat{B}_{\sigma(1)}(t_{\sigma(1)})\dots\hat{B}_{\sigma(n)}(t_{\sigma(n)}), \qquad \sigma\in\mathcal{S}_n: \quad t_{\sigma(1)}\geq\dots\geq t_{\sigma(n)}. \tag{11.41}$$

(Fermionic fields are freely anticommuted under the time ordering symbol T until in chronological order, when the T can be dropped.)

The equations (11.29) and (11.30) assume the same form even in the multicomponent case, but with the numbers $1,\dots,n$ representing not only the spacetime points $x_1,\dots,x_n$, but also the types of fields and their components. Moreover, each term carries a sign $(-1)^p$, where p is the number of transpositions of fermionic components needed to achieve the desired order of numbers $1,\dots,n$. Note that the propagators $\langle 12\rangle$ etc. are automatically zero when 1 and 2 correspond to different field components. To give an example,

$$T\big(\hat{\bar{\psi}}_\alpha(x_1)\hat{A}_\mu(x_2)\hat{\psi}_\beta(x_3)\big) = :\hat{\bar{\psi}}_\alpha(x_1)\hat{A}_\mu(x_2)\hat{\psi}_\beta(x_3): - i(S_F)_{\beta\alpha}(x_3-x_1)\ :\hat{A}_\mu(x_2):\ . \tag{11.42}$$

It is convenient to set up the following diagrammatic representation of the Feynman propagators of the Klein-Gordon field, Eq. (9.19), the Dirac field, Eq. (9.45), and the electromagnetic field, Eq. (10.30), respectively:



$$i\Delta_F(x-y) = \langle 0|\,T\big(\hat{\phi}(x)\hat{\phi}(y)\big)\,|0\rangle \qquad i(S_F)_{\alpha\beta}(x-y) = \langle 0|\,T\big(\hat{\psi}_\alpha(x)\hat{\bar{\psi}}_\beta(y)\big)\,|0\rangle \qquad i(D_F)_{\mu\nu}(x-y) = \langle 0|\,T\big(\hat{A}_\mu(x)\hat{A}_\nu(y)\big)\,|0\rangle \tag{11.43}$$

These are the basic building blocks of quantum field theoretical correlation functions.

11.3 Examples of interacting field theories

11.3.1 ϕ^4-theory

In the advanced quantum field theory course (02KTPA2) we will be dealing with *full* n-point correlation functions, i.e., the correlation functions of the full interacting theory. These will be given by

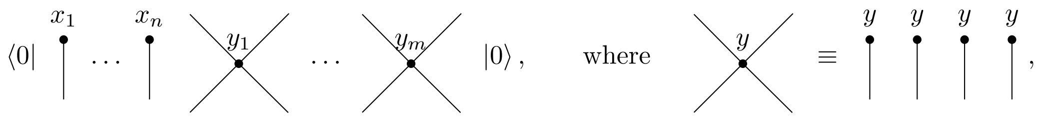
$$\frac{\langle 0|\,T\Big[\hat{\phi}(x_1)\dots\hat{\phi}(x_n)\exp\Big(-i\int_{-\infty}^{+\infty}dt\,\hat{H}_I^I(t)\Big)\Big]\,|0\rangle}{\langle 0|\,T\exp\Big(-i\int_{-\infty}^{+\infty}dt\,\hat{H}_I^I(t)\Big)\,|0\rangle}\ , \qquad \int_{-\infty}^{+\infty}dt\,\hat{H}_I^I(t) = \int d^4x\,\frac{\lambda}{4!}\hat{\phi}^4(x), \tag{11.44}$$

and reduce to (11.23) for vanishing interaction strength λ. (Here we consider, for simplicity, a single scalar field described by the Lagrangian (11.1) with a 'ϕ^4' interaction term.)

Perturbative calculation then features terms of the form

$$\frac{(-i\lambda)^m}{m!(4!)^m}\int d^4y_1\dots d^4y_m\ \langle 0|\,T\big(\hat{\phi}(x_1)\dots\hat{\phi}(x_n)\,\hat{\phi}^4(y_1)\dots\hat{\phi}^4(y_m)\big)\,|0\rangle\,, \tag{11.45}$$

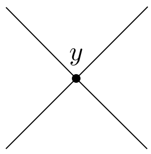
where the vacuum expectation values can be depicted diagrammatically as



(11.46)

and evaluated with a help of the Wick theorem, Eq. (11.37). According to this the dangling 'legs' of the 'prediagram' (11.46) get connected into edges in all possible ways, producing a sum of all graphs (*Feynman diagrams in the position space*) with m 4-valent vertices $y_1, \ldots, y_m$ (the 'interaction vertices'), and n single-valent vertices $x_1, \ldots, x_n$ (the 'external points').

The edges represent Feynman propagators $i\Delta_F$, and the interaction vertices, as a rule, implicitly carry the coupling constant, and are integrated over:



$$\equiv \quad -i\lambda \int d^4y \,, \tag{11.47}$$

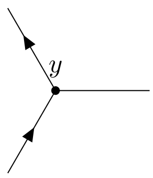
As an illustration of these diagrammatic techniques, we study in Exercise 43 the full two-point correlation function (the full propagator) of the ϕ^4-theory in first-order perturbation theory.

11.3.2 Yukawa interaction

A basic interaction that couples different kinds of fields (namely, a scalar boson with mass M and a fermion with mass m) is the Yukawa interaction, characterized by the last term in the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{M^2}{2}\phi^2 + \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \; \underbrace{-g\bar{\Psi}\Psi\phi}_{\mathscr{L}_I} . \tag{11.48}$$

The corresponding interaction vertex has three legs (one fermion, one anti-fermion, and one scalar):



$$\equiv \quad -ig \int d^4y. \tag{11.49}$$

We shall make use of the Yukawa interaction in examples of decay and scattering processes in Chapter 12.

11.3.3 Quantum electrodynamics

Classical electrodynamics describes the electromagnetic field and its interaction with (electrically charged) matter. In quantum electrodynamics, the electromagnetic field is quantized, and material particles (most typically, electrons with charge $q = -|e|$ and spin $\frac{1}{2}$) are excitations of a quantized Dirac field.

The interaction between the Dirac and electromagnetic field is introduced via minimal coupling, $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$ (see Section 4.1), yielding the 'Maxwell-Dirac' Lagrangian

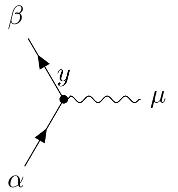
$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\big(i\gamma^\mu(\partial_\mu + iqA_\mu) - m\big)\Psi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \underbrace{-q\bar{\Psi}\gamma^\mu\Psi A_\mu}_{\mathscr{L}_I}. \tag{11.50}$$

(In quantum theory we also need to include the 'gauge fixing' term $-\frac{1}{2\xi}(\partial_\rho A^\rho)^2$ from Eq. 10.6.) Variations with respect to $\bar{\Psi}$ and A_μ produce the equations of motion

$$(i\gamma^\mu D_\mu - m)\Psi = 0 \qquad \text{and} \qquad \partial_\mu F^{\mu\nu} = q\bar{\Psi}\gamma^\nu\Psi, \tag{11.51}$$

respectively. Note that the Dirac current $q\bar{\Psi}\gamma^\mu\Psi$ figures on the right-hand side of the Maxwell equations as a source of the electromagnetic field, i.e., as an electromagnetic four-current j^μ.

The interaction term $\mathscr{L}_I$ is depicted by the vertex



$$\equiv \quad -iq(\gamma^\mu)_{\alpha\beta}\int d^4y. \tag{11.52}$$

11.3.4 Standard model interactions

In Section 10.4 we briefly reviewed the particle content of the Standard model. The particles' mutual interactions (or self-interactions in some cases) are embodied in various interaction terms of the Standard model Lagrangian. Without going into details, let us just note that all the previous examples of interaction vertices, (11.47), (11.49) and (11.52), play their role.

Other Standard model vertices are supplied by the non-Abelian (or Yang-Mills) gauge theories that describe gluons and gauge bosons, the mediators of the strong and weak interactions. In these theories, the "electromagnetic" four-potential is generalized to a matrix-valued object $\mathsf{A}_\mu = A^a_\mu \mathsf{T}_a$, where T_a are generators of a certain 'gauge group' of internal symmetries. The covariant derivative $D_\mu = \partial_\mu + ig\mathsf{A}_\mu$ defines a matrix-valued "Faraday tensor" (cf. Eq. (4.8))

$$\mathsf{F}_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu] = \partial_\mu\mathsf{A}_\nu - \partial_\nu\mathsf{A}_\mu + ig[\mathsf{A}_\mu, \mathsf{A}_\nu], \tag{11.53}$$

where the last term is present due to the non-commutativity of the matrices A_μ. The ensuing non-Abelian generalization of the Maxwell Lagrangian,

$$\mathscr{L} = -\frac{1}{4}\operatorname{Tr}(\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}), \tag{11.54}$$

contains third- and fourth-order terms, i.e., interactions between the gauge fields A^a_μ.

11.4 Exercises

Exercise 41. *Proof of the Wick theorem.* Show that for a free one-component real Klein-Gordon field $\hat{\phi}(x)$, and an arbitrary function $J(x)$,

$$T\exp\Big(i\int d^4x J(x)\hat{\phi}(x)\Big) = :\exp\Big(i\int d^4x J(x)\hat{\phi}(x)\Big): \ \exp\Big(-\frac{1}{2}\int d^4x d^4y J(x)\, i\Delta_F(x-y)\, J(y)\Big). \tag{11.55}$$

Solution:

We divide the proof into three steps.

1. Let us cast the left-hand side as

$$T\exp\Big(i\int d^4x J(x)\hat{\phi}(x)\Big) = \lim_{T\to\infty} T\exp\Big(-i\int_{-T}^{T} dt\, \hat{H}_I^I(t)\Big), \quad \hat{H}_I^I(t) = -\int d^3x J(\boldsymbol{x},t)\hat{\phi}(\boldsymbol{x},t), \tag{11.56}$$

and introduce a time slicing $(t_n)_{n=0}^N$ such that $t_0 = -T$, $t_N = T$, $N\Delta t = 2T$. It is convenient to use the representation (11.17) of the time-ordered exponential, and employ the restricted Baker-Campbell-Hausdorff formula $e^{\mathsf{A}}e^{\mathsf{B}} = e^{\mathsf{A}+\mathsf{B}}e^{\frac{1}{2}[\mathsf{A},\mathsf{B}]}$, Eq. (2.13), where for A and B we take $\hat{H}_I^I(t)$ at different times. (This is justified since $[\hat{\phi}(x),\hat{\phi}(y)] = i\Delta(x-y)$, the Pauli-Jordan function of Eq. (9.3), and hence $[\hat{H}_I^I(t),\hat{H}_I^I(t')] \in \mathbb{C}$.) We obtain

$$\begin{aligned}
T\exp\Big(-i\int_{-T}^{T} dt\, \hat{H}_I^I(t)\Big) &\approx e^{-i\Delta t\hat{H}_I^I(t_{N-1})}\dots e^{-i\Delta t\hat{H}_I^I(t_1)}e^{-i\Delta t\hat{H}_I^I(t_0)} \\
&= e^{-i\Delta t\hat{H}_I^I(t_{N-1})}\dots e^{-i\Delta t\hat{H}_I^I(t_2)}e^{-i\Delta t(\hat{H}_I^I(t_1)+\hat{H}_I^I(t_0))}e^{-\frac{1}{2}(\Delta t)^2[\hat{H}_I^I(t_1),\hat{H}_I^I(t_0)]} \\
&= e^{-i\Delta t\sum_{n=0}^{N-1}\hat{H}_I^I(t_n)}e^{-\frac{1}{2}\sum_{n>n'}(\Delta t)^2[\hat{H}_I^I(t_n),\hat{H}_I^I(t_{n'})]} \\
&\approx \exp\Big(-i\int_{-T}^{T} dt\, \hat{H}_I^I(t)\Big)\exp\Big(-\frac{1}{2}\int_{-T}^{T} dt\, dt'\, \theta(t-t')[\hat{H}_I^I(t),\hat{H}_I^I(t')]\Big),
\end{aligned} \tag{11.57}$$

that is,

$$\begin{aligned}
T\exp\Big(i\int d^4x\, J(x)\hat{\phi}(x)\Big) &= \exp\Big(i\int d^4x\, J(x)\hat{\phi}(x)\Big) \\
&\quad\times\exp\Big(-\frac{1}{2}\int d^4x\, d^4y\, J(x)J(y)\theta(x^0-y^0)[\hat{\phi}(x),\hat{\phi}(y)]\Big).
\end{aligned} \tag{11.58}$$

2. Now we will rewrite the normal-ordered part of Eq. (11.55). To this end we divide the field operator into two parts,

$$\hat{\phi}(x) = \int\frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}}\Big(\hat{a}_{\boldsymbol{p}}\, e^{-ip\cdot x} + \hat{a}_{\boldsymbol{p}}^\dagger\, e^{ip\cdot x}\Big) = \hat{\phi}^{(+)}(x) + \hat{\phi}^{(-)}(x), \tag{11.59}$$

the part $\hat{\phi}^{(+)}(x)$ containing all the annihilation operators, and the part $\hat{\phi}^{(-)}(x)$ containing all the creation operators. Normal ordering puts $\hat{\phi}^{(+)}$ to the right of $\hat{\phi}^{(-)}$, so

$$\begin{aligned}
:\exp\Big(i\int d^4x\, J(x)\hat{\phi}(x)\Big): &= \exp\Big(i\int d^4x\, J(x)\hat{\phi}^{(-)}(x)\Big)\exp\Big(i\int d^4x\, J(x)\hat{\phi}^{(+)}(x)\Big) \\
&= \exp\Big(i\int d^4x\, J(x)\hat{\phi}(x)\Big)\exp\Big(-\frac{1}{2}\int d^4x\, d^4y\, J(x)J(y)[\hat{\phi}^{(-)}(x),\hat{\phi}^{(+)}(y)]\Big),
\end{aligned} \tag{11.60}$$

where in the last step we used again the Baker-Campbell-Hausdorff identity. Combining expressions (11.58) and (11.60) yields

$$T\exp\Big(i\int d^4x\, J(x)\hat{\phi}(x)\Big) = :\exp\Big(i\int d^4x\, J(x)\hat{\phi}(x)\Big): \\ \times \exp\left(-\frac{1}{2}\int d^4x\, d^4y\, J(x)J(y)\Big(\theta(x^0-y^0)[\hat{\phi}(x),\hat{\phi}(y)] - [\hat{\phi}^{(-)}(x),\hat{\phi}^{(+)}(y)]\Big)\right). \tag{11.61}$$

3. To finish the proof we realize that, since $\hat{\phi}^{(+)}(x)\,|0\rangle = 0$ and $\langle 0|\,\hat{\phi}^{(-)}(x) = 0$,

$$\langle 0|\,[\hat{\phi}^{(-)}(x),\hat{\phi}^{(+)}(y)]\,|0\rangle = -\,\langle 0|\,\hat{\phi}^{(+)}(y)\hat{\phi}^{(-)}(x)\,|0\rangle = -\,\langle 0|\,\hat{\phi}(y)\hat{\phi}(x)\,|0\rangle\,, \tag{11.62}$$

and so the (number-valued) expression in the second exponential in Eq. (11.61) can be cast as

$$\begin{aligned}&\theta(x^0-y^0)[\hat{\phi}(x),\hat{\phi}(y)] - [\hat{\phi}^{(-)}(x),\hat{\phi}^{(+)}(y)]\\ &= \langle 0|\,\theta(x^0-y^0)\hat{\phi}(x)\hat{\phi}(y) - \theta(x^0-y^0)\hat{\phi}(y)\hat{\phi}(x) + \hat{\phi}(y)\hat{\phi}(x)\,|0\rangle\\ &= \langle 0|\,T\big(\hat{\phi}(x)\hat{\phi}(y)\big)\,|0\rangle\\ &= i\Delta_F(x-y),\end{aligned} \tag{11.63}$$

where we made use of the fact that $1-\theta(t) = \theta(-t)$.

Remarks:

Analysing the above proof one may easily ascertain that for a multicomponent bosonic field (ϕ_r) the Wick theorem extends to

$$T\exp\Big(i\int d^4x\, J_r(x)\hat{\phi}_r(x)\Big) = :\exp\Big(i\int d^4x\, J_r(x)\hat{\phi}_r(x)\Big): \\ \times \exp\Big(-\frac{1}{2}\int d^4x\, d^4y\, J_r(x)\,\langle 0|\,T\big(\hat{\phi}_r(x)\hat{\phi}_s(y)\big)\,|0\rangle\, J_s(y)\Big), \tag{11.64}$$

where there is now one Schwinger source J_r per each component of the field.

Exercise 42. *Four-point operator Wick expansion.* Using the Wick theorem (11.55), expand the time-ordered product

$$T\big(\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)\big) \tag{11.65}$$

in terms of normal-ordered products and Feynman propagators.

Solution:

We expand both sides of Eq. (11.55),

$$\begin{aligned}
&\frac{i^4}{4!}\int d^4x_1\dots d^4x_4\, J(x_1)\dots J(x_4)\, T\big(\hat\phi(x_1)\dots\hat\phi(x_4)\big)\\
&=\frac{i^4}{4!}\int d^4x_1\dots d^4x_4\, J(x_1)\dots J(x_4)\ \ {:}\hat\phi(x_1)\dots\hat\phi(x_4){:}\\
&+\frac{i^2}{2!}\left(-\frac{1}{2}\right)\int d^4x_1\dots d^4x_4\, J(x_1)\dots J(x_4)\ \ {:}\hat\phi(x_1)\hat\phi(x_2){:}\ \langle 0|\,T\big(\hat\phi(x_1)\hat\phi(x_2)\big)\,|0\rangle\\
&+\frac{1}{2!}\left(-\frac{1}{2}\right)^2\int d^4x_1\dots d^4x_4\, J(x_1)\dots J(x_4)\ \langle 0|\,T\big(\hat\phi(x_1)\hat\phi(x_2)\big)\,|0\rangle\,\langle 0|\,T\big(\hat\phi(x_3)\hat\phi(x_4)\big)\,|0\rangle\,,
\end{aligned}\tag{11.66}$$

and compare coefficients at $J(x_1)\dots J(x_4)$ (after symmetrizing over $x_1,\dots,x_4$). With the shorthand notation of Eq. (11.28) this gives

$$\frac{1}{4!}T(1234)=\frac{1}{4!}\,{:}1234{:}+\frac{1}{4!}\sum_{\sigma\in\mathcal{S}_4}\frac{1}{2!}\frac{1}{2}\,{:}\sigma(1)\sigma(2){:}\,\langle\sigma(3)\sigma(4)\rangle+\frac{1}{4!}\sum_{\sigma\in\mathcal{S}_4}\frac{1}{2!}\frac{1}{2^2}\langle\sigma(1)\sigma(2)\rangle\langle\sigma(3)\sigma(4)\rangle,\tag{11.67}$$

which simplifies to

$$\begin{aligned}
T(1234)={}&{:}1234{:}\\
&+{:}12{:}\,\langle 34\rangle+{:}13{:}\,\langle 24\rangle+{:}14{:}\,\langle 23\rangle+{:}23{:}\,\langle 14\rangle+{:}24{:}\,\langle 13\rangle+{:}34{:}\,\langle 12\rangle\\
&+\langle 12\rangle\langle 34\rangle+\langle 13\rangle\langle 24\rangle+\langle 14\rangle\langle 23\rangle.
\end{aligned}\tag{11.68}$$

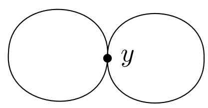
Exercise 43. *Full propagator of ϕ^4-theory in first order.* In ϕ^4-theory determine the full two-point function

$$\langle 12\rangle_{\text{full}}\equiv\frac{\langle 0|\,T\Big[\hat\phi(x_1)\hat\phi(x_2)\exp\Big(-\frac{i\lambda}{4!}\int d^4x\,\hat\phi^4(x)\Big)\Big]\,|0\rangle}{\langle 0|\,T\exp\Big(-\frac{i\lambda}{4!}\int d^4x\,\hat\phi^4(x)\Big)\,|0\rangle}\tag{11.69}$$

up to the order λ^1. (Use the Wick theorem and diagrammatic representation.)

Solution:

Consider first the denominator of Eq. (11.69) up to first order in λ:

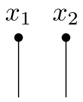
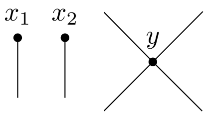
$$\mathcal{D}\equiv 1-\frac{i\lambda}{4!}\int d^4y\ \langle 0|\,\hat\phi^4(y)\,|0\rangle=1+\frac{3}{4!}$$ 

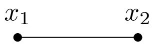
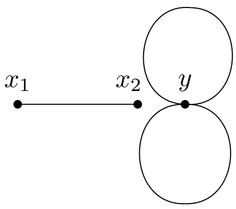
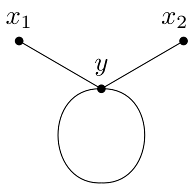
. (11.70)

Here the vertex (which is implicitly integrated over) carries a factor $-i\lambda$, and the factor 3 is the number of ways one can contract $\langle 0|\,\hat\phi^4(y)\,|0\rangle$ to a product of propagators, $\langle 0|\,\hat\phi^2(y)\,|0\rangle\,\langle 0|\,\hat\phi^2(y)\,|0\rangle$.

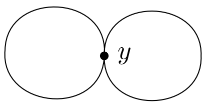
The numerator is, up to first order in λ, given by

$$\mathcal{N} \equiv \langle 0| T\big(\hat{\phi}(x_1)\hat{\phi}(x_2)\big) |0\rangle - \frac{i\lambda}{4!}\int d^4y \, \langle 0| T\big(\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}^4(y)\big) |0\rangle$$

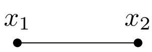
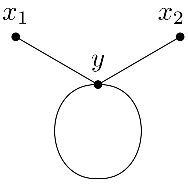
$= \langle 0|$  $|0\rangle - \frac{i\lambda}{4!}\int d^4y \, \langle 0|$  $|0\rangle$

$=$  $+ \frac{3}{4!}$  $+ \frac{4\cdot 3}{4!}$  . (11.71)

Finally, to determine $\langle 12\rangle_{\text{full}} = \mathcal{N}/\mathcal{D}$ up to order λ^1, we calculate

$\mathcal{D}^{-1} = 1 - \frac{3}{4!}$  $+ \mathcal{O}(\lambda^2),$ (11.72)

and

$\mathcal{N}\mathcal{D}^{-1} =$  $+ \frac{4\cdot 3}{4!}$  $+ \mathcal{O}(\lambda^2).$ (11.73)

In analytical terms, we have found that

$$\langle 12\rangle_{\text{full}} = i\Delta_F(x_1 - x_2) - \frac{i\lambda}{2}\int d^4y \, i\Delta_F(x_1 - y) i\Delta_F(0) i\Delta_F(y - x_2) + \mathcal{O}(\lambda^2). \tag{11.74}$$

Expressing the propagators in momentum space, according to formula (9.19), yields (up to the order λ^1)

$$\begin{aligned} \langle 12\rangle_{\text{full}} &= i\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x_1-x_2)}}{p^2 - m^2 + i\varepsilon} - \frac{i\lambda}{2} i\Delta_F(0)\, i^2 \int d^4y \frac{d^4p}{(2\pi)^4}\,\frac{d^4p'}{(2\pi)^4}\,\frac{e^{-ip\cdot(x_1-y)}}{p^2 - m^2 + i\varepsilon}\,\frac{e^{-ip'\cdot(y-x_2)}}{p'^2 - m^2 + i\varepsilon} \\ &= i\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x_1-x_2)}}{p^2 - m^2 + i\varepsilon} + \frac{i\lambda}{2} i\Delta_F(0)\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x_1-x_2)}}{(p^2 - m^2 + i\varepsilon)^2} \\ &= i\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x_1-x_2)}}{p^2 - m^2 + i\varepsilon}\left(1 + \frac{\lambda}{2}\frac{i\Delta_F(0)}{p^2 - m^2 + i\varepsilon}\right). \end{aligned} \tag{11.75}$$

For first order in λ we can now use the identity $1 + \lambda A \approx \frac{1}{1-\lambda A}$ to finally obtain

$$\langle 12\rangle_{\text{full}} = i\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x_1-x_2)}}{p^2 - m^2 - \frac{\lambda}{2} i\Delta_F(0) + i\varepsilon} + \mathcal{O}(\lambda^2). \tag{11.76}$$

That is, we see that the full propagator (up to the order λ^1) has the same form as the free propagator $i\Delta_F(x-y)$, only the mass gets shifted.

Remarks:

The mass shift (or correction) is determined by

$$i\Delta_F(0) = i\int \frac{dp_0\, d^3p}{(2\pi)^4}\, \frac{1}{p^2 - m^2 + i\varepsilon} = i\int_{-i\infty}^{i\infty} dp_0 \int \frac{d^3p}{(2\pi)^4}\, \frac{1}{p^2 - m^2}, \tag{11.77}$$

where we have rotated the integration contour of the variable p_0 from the real to imaginary axis. (According to Figure 9.1 this can be done without crossing the poles.) Substituting $p_0 = ip_4$ (where the variable p_4 now runs from $-\infty$ to $+\infty$), and transforming to four-dimensional spherical coordinates with $P = \sqrt{p_1^2 + \ldots + p_4^2}$,

$$i\Delta_F(0) = -\int \frac{dp_4\, d^3p}{(2\pi)^4}\, \frac{1}{p_4^2 + \boldsymbol{p}^2 + m^2} = -\int d\Omega \int_0^\infty \frac{dP}{(2\pi)^4}\, \frac{P^3}{P^2 + m^2}. \tag{11.78}$$

The latter integral over P, however, is clearly divergent, so, in order to keep calculations under control, one adopts a *regularization*, such as the momentum cut-off: $0 \le P \le \Lambda$.

The shifted mass is the one measured in experiments, hence it is called the *physical mass* (or *renormalized mass*) m_{ph}. We have the relation

$$m_{ph}^2 = m^2 + \frac{\lambda}{2} i\Delta_F(0;\Lambda). \tag{11.79}$$

Here m_{ph} is a finite value given by experiment, so $m(\Lambda)$ is viewed as a cut-off dependent quantity (the *bare mass*), which tends to infinity as $\Lambda \to \infty$.

In quantum field theory, infinities are rule rather than exception. They occur as a result of loops formed in Feynman diagrams. The *renormalization procedure* provides a systematic way to treat the infinities by absorbing them into ‘bare’ parameters of the Lagrangian. However, its success depends on the Lagrangian — some theories are renormalizable (the physically acceptable ones), while others are not renormalizable.

Chapter 12

Applications in particle physics

12.1 Decay of an unstable particle

Let us start with a concrete example. A scalar boson with mass M (e.g., the Higgs boson) decays into fermion and anti-fermion (with masses m). Let p_A denote the initial four-momentum of the boson, and let (p_1, s_1) and (p_2, s_2) be the four-momenta and spins of the fermion and the anti-fermion, respectively. Schematically, $A \to 1 + 2$. In view of Eqs. (7.60) and (8.27), we have the initial and final state

$$
\begin{aligned}
|i\rangle &= |\boldsymbol{p}_A\rangle = \sqrt{(2\pi)^3 2E_A}\,\hat{a}^\dagger_{\boldsymbol{p}_A}|0\rangle\,,\\
\langle f| &= \langle \boldsymbol{p}_1, s_1, \bar{\boldsymbol{p}}_2, \bar{s}_2| = \frac{\sqrt{(2\pi)^3 2E_1}}{\sqrt{2m}}\frac{\sqrt{(2\pi)^3 2E_2}}{\sqrt{2m}}\,\langle 0|\,\hat{d}_{\boldsymbol{p}_2,s_2}\hat{b}_{\boldsymbol{p}_1,s_1}\,,
\end{aligned}
\tag{12.1}
$$

where the energies are on-shell, i.e., $E_A = \sqrt{\boldsymbol{p}_A^2 + M^2}$ and $E_{1,2} = \sqrt{\boldsymbol{p}_{1,2}^2 + m^2}$.

Let us model the decay process by the Yukawa interaction (11.48), for which the interaction Hamiltonian reads, according to Eqs. (11.5) and (11.48),

$$
H_I = -\int d^3x\,\mathscr{L}_I = \int d^3x\, g\bar{\Psi}\Psi\phi, \tag{12.2}
$$

and calculate the amplitude

$$
S_{fi} = \langle f|\,T\mathrm{exp}\Big(-ig\int d^4x\,\hat{\bar{\Psi}}(x)\hat{\Psi}(x)\hat{\phi}(x)\Big)\,|i\rangle \approx -ig\int d^4x\,\langle f|\,\hat{\bar{\Psi}}(x)\hat{\Psi}(x)\hat{\phi}(x)\,|i\rangle \tag{12.3}
$$

in first order in the coupling constant g. Recall the mode expansions (Eqs. (7.14), (8.8) and (8.9))

$$
\begin{aligned}
\hat{\phi}(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2\Omega_{\boldsymbol{p}}}}\Big(\hat{a}_{\boldsymbol{p}}\,e^{-ip\cdot x} + \hat{a}^\dagger_{\boldsymbol{p}}\,e^{ip\cdot x}\Big) = \hat{\phi}^{(+)}(x) + \hat{\phi}^{(-)}(x),\\
\hat{\Psi}(x) &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{\omega_{\boldsymbol{p}}}}\Big(\hat{b}_{\boldsymbol{p},s}u(\boldsymbol{p},s)e^{-ip\cdot x} + \hat{d}^\dagger_{\boldsymbol{p},s}v(\boldsymbol{p},s)e^{ip\cdot x}\Big) = \hat{\Psi}^{(+)}(x) + \hat{\Psi}^{(-)}(x),\\
\hat{\bar{\Psi}}(x) &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{\omega_{\boldsymbol{p}}}}\Big(\hat{d}_{\boldsymbol{p},s}\bar{v}(\boldsymbol{p},s)e^{-ip\cdot x} + \hat{b}^\dagger_{\boldsymbol{p},s}\bar{u}(\boldsymbol{p},s)e^{ip\cdot x}\Big) = \hat{\bar{\Psi}}^{(+)}(x) + \hat{\bar{\Psi}}^{(-)}(x),
\end{aligned}
\tag{12.4}
$$

where we denote $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$, and $\Omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + M^2}$. Leaving out all the field products that do not pair up with the states (and so give automatically zero),

$$\langle f|\, \hat{\bar{\Psi}}(x)\hat{\Psi}(x)\hat{\phi}(x)\, |i\rangle = \langle \boldsymbol{p}_1, s_1, \bar{\boldsymbol{p}}_2, \bar{s}_2|\, \hat{\bar{\Psi}}^{(-)}(x)\hat{\Psi}^{(-)}(x)\hat{\phi}^{(+)}(x)\, |\boldsymbol{p}_A\rangle\,. \tag{12.5}$$

Now

$$\hat{\phi}^{(+)}(x)\, |\boldsymbol{p}_A\rangle = \int d^3p\, \frac{\sqrt{E_A}}{\sqrt{\Omega_{\boldsymbol{p}}}}\, e^{-ip\cdot x}\, \underbrace{\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}_A}\, |0\rangle}_{\delta(\boldsymbol{p}-\boldsymbol{p}_A)|0\rangle} = e^{-ip_A\cdot x}\, |0\rangle\,, \tag{12.6}$$

and for the fermionic fields acting on the 'bra' state:

$$\begin{aligned}
&\langle \boldsymbol{p}_1, s_1, \bar{\boldsymbol{p}}_2, \bar{s}_2|\hat{\bar{\Psi}}^{(-)}(x)\hat{\Psi}^{(-)}(x) \\
&= \sum_{s,s'} \int d^3p\, d^3p'\, \frac{\sqrt{E_1E_2}}{\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'}}}\, \underbrace{\langle 0|\, \hat{d}_{\boldsymbol{p}_2,s_2}\hat{b}_{\boldsymbol{p}_1,s_1}\hat{b}^\dagger_{\boldsymbol{p},s}\hat{d}^\dagger_{\boldsymbol{p}',s'}}_{\langle 0|\delta_{s_1 s}\delta(\boldsymbol{p}_1-\boldsymbol{p})\delta_{s_2,s'}\delta(\boldsymbol{p}_2-\boldsymbol{p}')}\ \bar{u}(\boldsymbol{p}, s)e^{ip\cdot x}v(\boldsymbol{p}', s')e^{ip'\cdot x} \\
&= \langle 0|\, e^{i(p_1+p_2)\cdot x}\bar{u}(\boldsymbol{p}_1, s_1)v(\boldsymbol{p}_2, s_2).
\end{aligned} \tag{12.7}$$

Thus, we find

$$\langle f|\, \hat{\bar{\Psi}}(x)\hat{\Psi}(x)\hat{\phi}(x)\, |i\rangle = \bar{u}(\boldsymbol{p}_1, s_1)v(\boldsymbol{p}_2, s_2)e^{i(p_1+p_2-p_A)\cdot x}. \tag{12.8}$$

The integral in Eq. (12.3) is easily done, and so we arrive at the decay amplitude in first perturbative order

$$S_{fi} \approx -ig\, \bar{u}(\boldsymbol{p}_1, s_1)v(\boldsymbol{p}_2, s_2)(2\pi)^4\delta(p_1 + p_2 - p_A). \tag{12.9}$$

With this example in mind, let us note that for any scattering (or decay) process, and in any order of perturbation theory, the scattering amplitude assumes the form

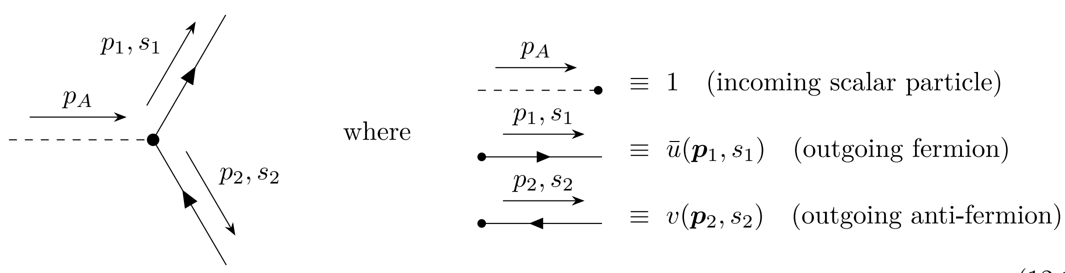
$$S_{fi} = \delta_{fi} + (2\pi)^4\delta(p_f - p_i)\, i\mathcal{M}_{fi}, \tag{12.10}$$

where $\mathcal{M}_{fi}$ is the so-called *invariant matrix element* (or *invariant amplitude*), p_i is the sum of four-momenta of particles in the initial state $|i\rangle$, and p_f is the sum of four-momenta of particles in the final state $|f\rangle$. The delta function expresses conservation of total four-momentum, which is a consequence of the Lagrangian not depending explicitly on x.

In our decay example, Eq. (12.9),

$$i\mathcal{M}_{fi} \approx -ig\, \bar{u}(\boldsymbol{p}_1, s_1)v(\boldsymbol{p}_2, s_2), \tag{12.11}$$

and we may represent this object diagrammatically as



(12.12)

and where the interaction vertex carries a factor $-ig$. In fact, general rules (the *Feynman rules for S matrix*) can be developed that allow one to represent the invariant matrix element $\mathcal{M}_{fi}$ graphically in any order of the perturbation theory, effectively bypassing the somewhat lengthy derivations based on mode expansions of the field operators. They will be systematically presented in the advanced course (02KTPA2).

To obtain the *probability* of a given process we need to take the modulus squared of the scattering amplitude (12.10). Assuming $i \neq f$,

$$|S_{fi}|^2 = (2\pi)^8 \underbrace{\delta(p_f - p_i)}_{\delta(p=0)} \delta(p_f - p_i)|\mathcal{M}_{fi}|^2 = (2\pi)^4 VT\, \delta(p_f - p_i)|\mathcal{M}_{fi}|^2, \tag{12.13}$$

where V is the spatial volume, and T the duration of the process (both tend to infinity), and they stem from the formula

$$\delta(p = 0) = \int \frac{d^4x}{(2\pi)^4}\, e^{ip\cdot x}|_{p=0} = \frac{VT}{(2\pi)^4}, \tag{12.14}$$

similar to Eq. (7.23). The expression (12.13) has to be normalized by (recall Eqs. (7.61) and (8.28))

$$\langle \boldsymbol{p}|\boldsymbol{p}\rangle = 2E_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\boldsymbol{0}) = 2E_{\boldsymbol{p}}V \qquad \text{or} \qquad \langle \boldsymbol{p}, s|\boldsymbol{p}, s\rangle = \frac{2E_{\boldsymbol{p}}V}{2m}, \tag{12.15}$$

respectively, for each bosons or fermions with mass m.

The (differential) probability is obtained by multiplying $\frac{|S_{fi}|^2}{\langle i|i\rangle\langle f|f\rangle}$ by the number of states in an infinitesimal neighbourhood of a given final state. For one particle this is given by $\frac{V d^3p}{(2\pi)^3}$, since the momentum of a particle in a box of volume $V = L^3$ is, by Eq. (7.24), $\boldsymbol{p} = \frac{2\pi}{L}\boldsymbol{k}$, where $k \in \mathbb{Z}^3$ a discrete mode label.

For a process $A \to 1 + \ldots + n$ with one initial decaying particle, the (differential) decay rate can be calculated as

$$\begin{aligned}
d\Gamma &= \frac{\text{differential probability of the process}}{\text{time}} \\
&= \frac{1}{T}\,\frac{V d^3p_1}{(2\pi)^3}\cdots\frac{V d^3p_n}{(2\pi)^3}\,\frac{|S_{fi}|^2}{\langle i|i\rangle\langle f|f\rangle} \\
&= \frac{N_A}{2E_A}\,\frac{N_1\, d^3p_1}{(2\pi)^3 2E_1}\cdots\frac{N_n\, d^3p_n}{(2\pi)^3 2E_n}\,(2\pi)^4\delta(p_1 + \ldots + p_n - p_A)\,|\mathcal{M}_{fi}|^2,
\end{aligned} \tag{12.16}$$

where $N = 1$ for a boson, and $N = 2m$ for a fermion with mass m. (The latter factors are due to our normalization conventions for the polarization spinors u and v — see Eq. (3.30).) We also note that if the final state contains indistinguishable particles, factors $\frac{1}{n_k!}$, where n_k counts the multiplicity of particles of species k, have to be included.

Consider for definiteness the initial particle at rest, $p_A = (m_A, \boldsymbol{0})$, and only two decay products with equal masses $m_1 = m_2$. Integrating $d\Gamma$ over the final momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ (and summing over final spins) provides the total decay rate Γ and the mean lifetime $1/\Gamma$ of the particle A. To this end we need to calculate the integral

$$\begin{aligned}
\mathcal{I} &\equiv \int \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\,(2\pi)^4\delta(E_1 + E_2 - m_A)\delta(\boldsymbol{p}_1 + \boldsymbol{p}_2)\,|\mathcal{M}_{fi}|^2 \\
&= \int \frac{d^3p_1}{(2\pi)^2 4E_1^2}\,\delta(2E_1 - m_A)\,|\mathcal{M}_{fi}|^2 \\
&= \int d\Omega \int_0^\infty \frac{dP\, P^2}{(2\pi)^2}\,\frac{\delta(2\sqrt{P^2 + m_1^2} - m_A)}{4(P^2 + m_1^2)}\,|\mathcal{M}_{fi}|^2, \qquad (P \equiv |\boldsymbol{p}_1|).
\end{aligned} \tag{12.17}$$

The last δ-function can be eliminated if we identify

$$f(P) \equiv 2\sqrt{P^2 + m_1^2} - m_A : \quad f(P_0) = 0 \text{ for } P_0 = \sqrt{\frac{m_A^2}{4} - m_1^2}, \quad f'(P_0) = \frac{2P_0}{\sqrt{P_0^2 + m_1^2}}, \tag{12.18}$$

and use the formula (7.67), $\delta(f(P)) = \frac{\delta(P-P_0)}{|f'(P_0)|}$:

$$\mathcal{I} = \int d\Omega \int_0^\infty \frac{dP\, P_0^2}{(2\pi)^2} \frac{\delta(P - P_0)}{8P_0\sqrt{P_0^2 + m_1^2}} |\mathcal{M}_{fi}|^2 = \int d\Omega \frac{1}{32\pi^2} \frac{\sqrt{m_A^2 - 4m_1^2}}{m_A} |\mathcal{M}_{fi}|^2. \tag{12.19}$$

The total (integrated) decay rate $\Gamma = \frac{N_A}{2m_A} N_1 N_2 \mathcal{I}$ reads

$$\Gamma = \int d\Omega \frac{d\Gamma}{d\Omega}, \quad \text{where} \quad \frac{d\Gamma}{d\Omega} = \frac{N_A N_1 N_2}{64\pi^2 m_A^2} \sqrt{m_A^2 - 4m_1^2}\, |\mathcal{M}_{fi}|^2. \tag{12.20}$$

Here, the quantity under the square root must be non-negative, i.e., $m_A \geq 2m_1$ — the rest mass of the decaying particle must be greater than the sum of the rest masses of the products.

In Exercise 44 we calculate the total decay rate Γ, including the spin summation, for the Yukawa theory decay treated at the beginning of this section.

12.2 Scattering cross section

Let us consider scattering processes $A + B \to 1 + \ldots + n$ involving two particles in the initial state. (In reality two beams of particles, one of which is viewed as the incoming beam and the other as the target.)

The probability of a given process per unit time is proportional to the incoming flux $\frac{1}{V}|\boldsymbol{v}_A - \boldsymbol{v}_B|$ (one particle density times "relative" velocity). The proportionality factor is the (differential) *scattering cross section*

$$\begin{aligned} d\sigma &= \frac{\text{differential probability of the process}}{(\text{time}) \times (\text{incident flux})} \\ &= \frac{1}{T} \frac{V}{|\boldsymbol{v}_A - \boldsymbol{v}_B|} \frac{V d^3p_1}{(2\pi)^3} \cdots \frac{V d^3p_n}{(2\pi)^3} \frac{|S_{fi}|^2}{\langle i|i\rangle\langle f|f\rangle} \\ &= \frac{N_A N_B}{2E_A 2E_B |\boldsymbol{v}_A - \boldsymbol{v}_B|} \frac{N_1\, d^3p_1}{(2\pi)^3 2E_1} \cdots \frac{N_n\, d^3p_n}{(2\pi)^3 2E_n} (2\pi)^4 \delta(p_1 + \ldots + p_n - p_A - p_B)\, |\mathcal{M}_{fi}|^2. \end{aligned} \tag{12.21}$$

It measures the likelihood of the given interaction between A and B in units of area, so it can be visualized as an effective area of the target particle. However, it also depends on the range of final states considered. (Ultimately, by integrating over all momenta and spins of the final state particles we obtain the total cross section σ.)

From now on we will focus on processes $A + B \to 1 + 2$ involving only two outgoing particles. Based on the conservation law $p_A + p_B = p_1 + p_2$ one defines the (Lorentz-invariant) *Mandelstam invariants*

$$\begin{aligned} s &= (p_A + p_B)^2 = (p_2 + p_1)^2 = 2p_A \cdot p_B + m_A^2 + m_B^2, \\ t &= (p_A - p_1)^2 = (p_B - p_2)^2 = -2p_A \cdot p_1 + m_A^2 + m_1^2, \\ u &= (p_A - p_2)^2 = (p_B - p_1)^2 = -2p_A \cdot p_2 + m_A^2 + m_2^2, \end{aligned} \tag{12.22}$$

which satisfy

$$s + t + u = m_A^2 + m_B^2 + m_1^2 + m_2^2. \tag{12.23}$$

In the center-of-mass frame, in which $\boldsymbol{p}_B = -\boldsymbol{p}_A$ (and, by conservation of momentum, also $\boldsymbol{p}_2 = -\boldsymbol{p}_1$), we have

$$p_A + p_B = (E_A + E_B, \mathbf{0}) \quad \to \quad s = (E_A + E_B)^2, \tag{12.24}$$

so $\sqrt{s}$ is the total energy in the center-of-mass frame. (The other two Mandelstam invariants t and u are related to the scattering angle between three-momenta $\boldsymbol{p}_A$ and $\boldsymbol{p}_1$.)

In addition to working in the center-of-mass frame, let us assume, for simplicity, equal masses: $m_A = m_B = m_1 = m_2 = m$. In this case $E_A = E_B = \frac{\sqrt{s}}{2}$, and since $\boldsymbol{v} = \frac{\boldsymbol{p}}{E}$,

$$E_A E_B |\boldsymbol{v}_A - \boldsymbol{v}_B| = |E_B \boldsymbol{p}_A - E_A \boldsymbol{p}_B| = 2E_A\,|\boldsymbol{p}_A| = \sqrt{s}\,\frac{\sqrt{s-4m^2}}{2}. \tag{12.25}$$

Integration over final-state momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ in Eq. (12.21) involves the same integral as in Eqs. (12.17) and (12.19), only with the replacement $m_A \to \sqrt{s}$:

$$\begin{aligned} \mathcal{I}|_{m_A \to \sqrt{s}} &= \int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta(E_1 + E_2 - \sqrt{s}) \delta(\boldsymbol{p}_1 + \boldsymbol{p}_2)\, |\mathcal{M}_{fi}|^2 \\ &= \int d\Omega\, \frac{1}{32\pi^2} \frac{\sqrt{s-4m^2}}{\sqrt{s}}\, |\mathcal{M}_{fi}|^2. \end{aligned} \tag{12.26}$$

Combining Eqs. (12.21), (12.25) and (12.26), in the case of equal masses the scattering cross section in the center-of-mass frame is given by

$$\sigma = \int d\Omega\, \frac{d\sigma}{d\Omega}, \quad \text{with} \quad \frac{d\sigma}{d\Omega} = \frac{N_A N_B}{2E_A 2E_B |\boldsymbol{v}_A - \boldsymbol{v}_B|} \frac{N_1 N_2}{32\pi^2} \frac{\sqrt{s-4m^2}}{\sqrt{s}}\, |\mathcal{M}_{fi}|^2 = \frac{N_A N_B N_1 N_2}{64\pi^2 s}\, |\mathcal{M}_{fi}|^2. \tag{12.27}$$

Let us emphasise that the quantity $|\mathcal{M}_{fi}|^2$ is subject to all constraints that we have gathered during our derivation: $\boldsymbol{p}_2 = -\boldsymbol{p}_1$, $E_1 = E_2 = E_A = E_B = \frac{\sqrt{s}}{2}$, and $|\boldsymbol{p}_1| = \sqrt{\frac{s}{4} - m^2}$.

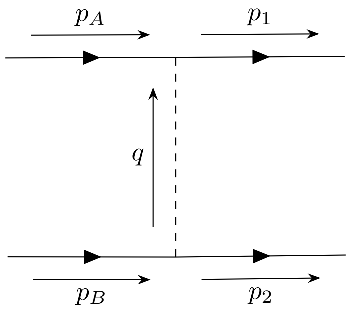
The initial and final states are characterized not only by the particle's momenta, but also, possibly, by spins. If not measuring the spins, one should *average* over the spin states of the initial particles, and *sum* over the spin states of the final state particles.

12.2.1 Yukawa potential

Yukawa interaction $\mathscr{L}_I = -g\bar{\Psi}\Psi\phi$ is capable of describing a scattering $A + B \to 1 + 2$ of two fermions (of masses m) mediated by a scalar boson (with mass M). In Exercise 45 we identify the invariant matrix element in second-order perturbation theory as

$$i\mathcal{M}_{fi} \approx (-ig)^2\, \bar{u}(1)u(A)\, \frac{i}{(p_1 - p_A)^2 - M^2} \bar{u}(2)u(B)\; - (1 \leftrightarrow 2), \quad \text{where} \quad \bar{u}(1) = \bar{u}(p_1, s_1), \text{ etc.}. \tag{12.28}$$

The first term is represented diagrammatically as



(12.29)

The internal line represents the Fourier transform of the Feynman propagator $i\Delta_F$, whose four-momentum $q = p_1 - p_A$ is 'off-shell' (it need not satisfy the relativistic energy-momentum dispersion relation $q^2 = M^2$). The boson is referred to as a 'virtual particle'.

In non-relativistic limit $|\boldsymbol{p}| \ll m$,

$$\omega_{\boldsymbol{p}} \approx m \quad , \quad p \approx (m, \boldsymbol{p}), \quad \text{and} \quad u(\boldsymbol{p}, s) = \frac{\sqrt{\omega_{\boldsymbol{p}} + m}}{\sqrt{2m}} \begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\omega_{\boldsymbol{p}}+m}\chi_s \end{pmatrix} \approx \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} = u(\boldsymbol{0}, s) \tag{12.30}$$

according to Eq. (3.22). Hence,

$$\bar{u}(\boldsymbol{p}_1, s_1)u(\boldsymbol{p}_A, s_A) \approx \bar{u}(\boldsymbol{0}, s_1)u(\boldsymbol{0}, s_A) = \delta_{s_1,s_A} \quad , \quad \bar{u}(\boldsymbol{p}_2, s_2)u(\boldsymbol{p}_B, s_B) \approx \delta_{s_2,s_B}, \tag{12.31}$$

which together with $(p_1 - p_A)^2 \approx -(\boldsymbol{p}_1 - \boldsymbol{p}_A)^2$ reduces Eq. (12.28) to

$$i\mathcal{M}_{fi}^{(NR)} \approx (-ig)^2 \frac{-i\,\delta_{s_1,s_A}\,\delta_{s_2,s_B}}{(\boldsymbol{p}_1 - \boldsymbol{p}_A)^2 + M^2} - (1 \leftrightarrow 2). \tag{12.32}$$

In the non-relativistic limit the Yukawa interaction conserves the spin degree of freedom. The second ('exchange') term $(1 \leftrightarrow 2)$ accounts for indistinguishability of the particles involved, and the minus sign is a consequence of their fermionic statistics.

Non-relativistic scattering of two particles can be also described within ordinary quantum mechanics. There one considers a two-particle Hamiltonian

$$\hat{H} = \underbrace{\frac{\hat{\boldsymbol{p}}_1^2}{2m} + \frac{\hat{\boldsymbol{p}}_2^2}{2m}}_{\hat{H}_0} + \hat{V}, \tag{12.33}$$

consisting of the 'free' part (kinetic energies of the particles), and an interparticle interaction potential $\hat{V} = V(\hat{\boldsymbol{x}}_1 - \hat{\boldsymbol{x}}_2)$. The initial and final states of the two fermions are antisymmetric,

$$|i\rangle = \frac{1}{\sqrt{2}}\big(\,|\boldsymbol{p}_A, s_A, \boldsymbol{p}_B, s_B\rangle - (A \leftrightarrow B)\big) \quad , \quad |f\rangle = \frac{1}{\sqrt{2}}\big(\,|\boldsymbol{p}_1, s_1, \boldsymbol{p}_2, s_2\rangle - (1 \leftrightarrow 2)\big), \tag{12.34}$$

where $\langle \boldsymbol{x}|\boldsymbol{p}\rangle = e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$, so that $\langle \boldsymbol{p}, s|\boldsymbol{p}', s'\rangle = (2\pi)^3\delta_{s,s'}\delta(\boldsymbol{p} - \boldsymbol{p}')$ coincides with the normalization (8.28) in non-relativistic limit.

The Dyson series (11.18) (with $\hat{H}_I = V$) provides the scattering amplitude (in the lowest order in the interaction potential)

$$S_{fi}^{(QM)} = \langle f|\,T\text{exp}\Big(-i\int_{-\infty}^{+\infty} dt\,\hat{V}^I(t)\Big)\,|i\rangle \approx -i\int_{-\infty}^{+\infty} dt\,\,\langle f|\,\hat{V}^I(t)\,|i\rangle\,, \tag{12.35}$$

where we have assumed that $\langle f|i\rangle = 0$. According to Eq. (11.8) we have $\hat{V}^I(t) = e^{it\hat{H}_0^S}\hat{V}\,e^{-it\hat{H}_0^S}$, and hence

$$S_{fi}^{(QM)} \approx -i\int_{-\infty}^{+\infty} dt\, e^{it\left(\frac{\boldsymbol{p}_1^2}{2m}+\frac{\boldsymbol{p}_2^2}{2m}\right)}\,\langle f|\,\hat{V}\,|i\rangle\, e^{-it\left(\frac{\boldsymbol{p}_A^2}{2m}+\frac{\boldsymbol{p}_B^2}{2m}\right)}$$
$$= -i\,2\pi\delta(E_f-E_i)\,\frac{1}{2}\Big(\delta_{s_1 s_A}\delta_{s_2 s_B}\langle \boldsymbol{p}_1,\boldsymbol{p}_2|\,\hat{V}\,|\boldsymbol{p}_A,\boldsymbol{p}_B\rangle - (A\leftrightarrow B) - (1\leftrightarrow 2) + (A\leftrightarrow B, 1\leftrightarrow 2)\Big), \tag{12.36}$$

where we have denoted $E_f \equiv \frac{\boldsymbol{p}_1^2}{2m}+\frac{\boldsymbol{p}_2^2}{2m}$ and $E_i \equiv \frac{\boldsymbol{p}_A^2}{2m}+\frac{\boldsymbol{p}_B^2}{2m}$, and assumed that the interaction potential does not involve any spin interactions. Furthermore, we employ spatial resolution of unity $\int d^3x_1 d^3x_2\,|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle\langle\boldsymbol{x}_1,\boldsymbol{x}_2| = \hat{1}$ to calculate

$$\langle \boldsymbol{p}_1,\boldsymbol{p}_2|\,\hat{V}\,|\boldsymbol{p}_A,\boldsymbol{p}_B\rangle = \int d^3x_1 d^3x_2\, V(\boldsymbol{x}_1-\boldsymbol{x}_2) e^{-i(\boldsymbol{p}_1\cdot\boldsymbol{x}_1+\boldsymbol{p}_2\cdot\boldsymbol{x}_2)} e^{i(\boldsymbol{p}_A\cdot\boldsymbol{x}_1+\boldsymbol{p}_B\cdot\boldsymbol{x}_2)}$$
$$= \int d^3x_1 d^3x_2\, V(\boldsymbol{x}_1-\boldsymbol{x}_2) e^{-i(\boldsymbol{p}_1-\boldsymbol{p}_A)\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)} e^{-i(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{p}_A-\boldsymbol{p}_B)\cdot\boldsymbol{x}_2}, \tag{12.37}$$

and the change of integration variables $\boldsymbol{x} = \boldsymbol{x}_1-\boldsymbol{x}_2$, $\boldsymbol{y} = \boldsymbol{x}_2$, $d^3x_1 d^3x_2 = d^3x\, d^3y$ to obtain

$$\langle \boldsymbol{p}_1,\boldsymbol{p}_2|\,\hat{V}\,|\boldsymbol{p}_A,\boldsymbol{p}_B\rangle = (2\pi)^3\delta^{(3)}(\boldsymbol{p}_f-\boldsymbol{p}_i)\,\widetilde{V}(\boldsymbol{p}_1-\boldsymbol{p}_A), \quad \text{where} \quad \widetilde{V}(\boldsymbol{q}) = \int d^3x\, V(\boldsymbol{x}) e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}. \tag{12.38}$$

Altogether we find the scattering amplitude

$$S_{fi}^{(QM)} \approx -i\,(2\pi)^4\delta^{(4)}(p_f-p_i)\,\frac{1}{2}\Big(\delta_{s_1,s_A}\delta_{s_2,s_B}\widetilde{V}(\boldsymbol{p}_1-\boldsymbol{p}_A) - (A\leftrightarrow B) - (1\leftrightarrow 2) + (A\leftrightarrow B, 1\leftrightarrow 2)\Big) \tag{12.39}$$

Here the δ-function implies the relation $p_1+p_2 = p_A+p_B$, which renders the first term equal to the last term, and the second term equal to the third term. The invariant matrix element is then identified with a help of Eq. (12.10) as

$$i\mathcal{M}_{fi}^{(QM)} \approx (-i)\delta_{s_1,s_A}\delta_{s_2,s_B}\widetilde{V}(\boldsymbol{p}_1-\boldsymbol{p}_A) - (1\leftrightarrow 2). \tag{12.40}$$

Equating the latter with the quantum field theoretic formula (12.32), i.e., $\mathcal{M}_{fi}^{(QM)} = \mathcal{M}_{fi}^{(NR)}$, leads to the identification

$$\widetilde{V}(\boldsymbol{q}) = -\frac{g^2}{\boldsymbol{q}^2+M^2} \qquad \to \qquad V(r) = -\frac{g^2}{4\pi}\frac{e^{-Mr}}{r} \qquad (r\equiv|\boldsymbol{x}|), \tag{12.41}$$

where the last step — inversion of the Fourier transform from $\widetilde{V}(\boldsymbol{q})$ to $V(r)$ — is carried out in Exercise 46. The function $V(r)$ is called the *Yukawa potential*.

This potential is attractive and 'short-ranged' — it effectively vanishes for distances $r \gg \frac{\hbar}{Mc}$ (the Compton wavelength of the boson). For distance 10^{-15} m (roughly the size of a nucleon) the corresponding rest energy of the mediating scalar particle is found as

$$Mc^2 = \frac{\hbar c}{10^{-15}\,\mathrm{m}} \doteq \frac{10^{-34}\cdot 3\cdot 10^8}{10^{-15}}\,\mathrm{J} \doteq 3\cdot 10^{-11}\,\frac{10^{13}}{1.6}\,\mathrm{MeV} \doteq 200\,\mathrm{MeV}. \tag{12.42}$$

This is in rough agreement with the rest energy of the mediator of the nuclear force, the π^0-meson: 135 MeV.

For a scattering of fermion and anti-fermion the Yukawa potential turns out to be again attractive, as well as for a scattering of anti-fermion and anti-fermion.

In a similar way one can derive the Coulomb potential starting from quantum electrodynamics and considering a scattering of two electrically charged particles [4, Ch. 4.8]. The interaction turns out to be attractive for opposite charges (a particle and an anti-particle), and repulsive for like charges, in agreement with classical electrodynamics. Since the mediator, the photon, is massless, the factor e^{-Mr} is not present, and one recovers the usual form of the Coulomb potential $V(r) \propto \frac{1}{r}$, which is 'long-ranged'.

Instead of forces and potentials between particles, relativistic quantum field theory offers a more fundamental picture where interactions happen due to an exchange of intermediate 'virtual' particles (bosons).

12.3 Exercises

Exercise 44. *Decay in Yukawa theory.* The invariant matrix element of a process $A \to 1 + 2$ in the first order of Yukawa theory is given by

$$i\mathcal{M}_{fi} \approx -ig\,\bar{u}(\boldsymbol{p}_1, s_1)v(\boldsymbol{p}_2, s_2) \tag{12.43}$$

(Eq. (12.11)). Determine the total (i.e., integrated over the momenta, and summed over the spins of the outgoing particles) decay rate Γ in the rest frame of the decaying particle.

Solution:

The total decay rate is given by spin summation of the formula (12.20) (with $N_A = 1$, $N_1 = N_2 = 2m$):

$$\Gamma = \sum_{s_1, s_2} \int d\Omega\, \frac{(2m)^2}{64\pi^2 m_A^2} \sqrt{m_A^2 - 4m_1^2}\, |\mathcal{M}_{fi}|^2. \tag{12.44}$$

Here

$$\begin{aligned} |\mathcal{M}_{fi}|^2 &= g^2\, \bar{u}(\boldsymbol{p}_1, s_1)v(\boldsymbol{p}_2, s_2)\left(\bar{u}(\boldsymbol{p}_1, s_1)v(\boldsymbol{p}_2, s_2)\right)^\dagger \\ &= g^2\, \bar{u}(\boldsymbol{p}_1, s_1)v(\boldsymbol{p}_2, s_2)\,\bar{v}(\boldsymbol{p}_2, s_2)u(\boldsymbol{p}_1, s_1) \\ &= g^2\, \mathrm{Tr}\Big(u(\boldsymbol{p}_1, s_1)\bar{u}(\boldsymbol{p}_1, s_1)\, v(\boldsymbol{p}_2, s_2)\bar{v}(\boldsymbol{p}_2, s_2)\Big). \end{aligned} \tag{12.45}$$

The spin sums of Eq. (9.42),

$$\sum_s u(\boldsymbol{p}, s)\bar{u}(\boldsymbol{p}, s) = \frac{\not{p} + m}{2m} \quad , \quad \sum_s v(\boldsymbol{p}, s)\bar{v}(\boldsymbol{p}, s) = \frac{\not{p} - m}{2m}, \tag{12.46}$$

allow us to simplify

$$\sum_{s_1, s_2} |\mathcal{M}_{fi}|^2 = \frac{g^2}{(2m)^2}\, \mathrm{Tr}\Big((\not{p}_1 + m)(\not{p}_2 - m)\Big) = \frac{g^2}{(2m)^2}(4p_1 \cdot p_2 - 4m^2), \tag{12.47}$$

where we have used the 'trace' identities (2.73) and (2.75).

Moreover, by total four-momentum conservation, $p_1 + p_2 = p_A = (M, \mathbf{0})$,

$$2p_1 \cdot p_2 = (p_1 + p_2)^2 - p_1^2 - p_2^2 = M^2 - 2m^2, \tag{12.48}$$

where $m_A = M$ is the mass of the decaying scalar particle, and $m_1 = m_2 = m$ is the mass of the products. Hence,

$$\sum_{s_1, s_2} |\mathcal{M}_{fi}|^2 = \frac{2g^2}{(2m)^2}(M^2 - 4m^2), \tag{12.49}$$

and plugging this into Eq. (12.44) yields

$$\Gamma = \frac{(2m)^2}{16\pi M^2} \sqrt{M^2 - 4m^2}\, \frac{2g^2}{(2m)^2}(M^2 - 4m^2) = \frac{g^2}{8\pi M^2}(M^2 - 4m^2)^{3/2}. \tag{12.50}$$

Exercise 45. *Scattering in Yukawa theory.* Consider a scattering process $A+B \to 1+2$ of spin-$\frac{1}{2}$ fermions with all masses equal ($f + f \to f + f$). Determine the invariant matrix element $i\mathcal{M}_{fi}$ in the lowest nontrivial order of the Yukawa theory. (Assume that the states of the incoming particles differ from the states of the outgoing particles.)

Solution:

We have the initial and final state

$$|i\rangle = |\boldsymbol{p}_A, s_A, \boldsymbol{p}_B, s_B\rangle = \frac{\sqrt{(2\pi)^3 2E_A}}{\sqrt{2m}} \frac{\sqrt{(2\pi)^3 2E_B}}{\sqrt{2m}} \, \hat{b}^\dagger_{\boldsymbol{p}_A, s_A} \hat{b}^\dagger_{\boldsymbol{p}_B, s_B} |0\rangle \,,$$
$$\langle f| = \langle \boldsymbol{p}_1, s_1, \boldsymbol{p}_2, s_2| = \frac{\sqrt{(2\pi)^3 2E_1}}{\sqrt{2m}} \frac{\sqrt{(2\pi)^3 2E_2}}{\sqrt{2m}} \, \langle 0| \, \hat{b}_{\boldsymbol{p}_2, s_2} \hat{b}_{\boldsymbol{p}_1, s_1} \,, \tag{12.51}$$

where m denotes the fermions' mass, and all the operators $\hat{b}$, $\hat{b}^\dagger$ mutually commute as they correspond (by assumption) to different states. The scattering amplitude for the case of Yukawa interaction $\mathscr{L}_I = -g\bar{\Psi}\Psi\phi$ is

$$S_{fi} = \langle f| \, T \exp\Big(-ig \int d^4x \, \hat{\bar{\Psi}}(x) \hat{\Psi}(x) \hat{\phi}(x) \Big) \, |i\rangle \,. \tag{12.52}$$

The lowest nontrivial order (in which the creation and annihilation operators defining the initial and final state are 'paired off' with those supplied by the mode expansions of the fields) is the second order in g:

$$S_{fi} \approx \frac{(-ig)^2}{2!} \int d^4x \, d^4y \, \langle f| \, T\Big(\hat{\bar{\Psi}}(x) \hat{\Psi}(x) \hat{\phi}(x) \hat{\bar{\Psi}}(y) \hat{\Psi}(y) \hat{\phi}(y) \Big) \, |i\rangle \,. \tag{12.53}$$

Employing the Wick expansion (11.29), generalized to multiple fields (see the comments in Section 11.2.2), the only term that survives taking $\langle f| \ldots |i\rangle$ is the one with scalar field propagator:

$$S_{fi} \approx \frac{(-ig)^2}{2!} \int d^4x \, d^4y \, \langle f| \, :\hat{\bar{\Psi}}(x) \hat{\Psi}(x) \hat{\bar{\Psi}}(y) \hat{\Psi}(y): \, |i\rangle \, \langle 0| \, T\big(\hat{\phi}(x) \hat{\phi}(y) \big) \, |0\rangle \,. \tag{12.54}$$

Now recall the mode expansions (12.4) of the fermionic field operators $\hat{\Psi}$ and $\hat{\bar{\Psi}}$, and retain only the parts $\hat{\Psi}^{(+)}$ and $\hat{\bar{\Psi}}^{(-)}$, containing $\hat{b}$ and $\hat{b}^\dagger$, as these 'pair' with the states (12.51):

$$\langle f| \, :\hat{\bar{\Psi}}(x) \hat{\Psi}(x) \hat{\bar{\Psi}}(y) \hat{\Psi}(y): \, |i\rangle = \langle f| \, :\hat{\bar{\psi}}^{(-)}_\alpha(x) \hat{\psi}^{(+)}_\alpha(x) \hat{\bar{\psi}}^{(-)}_\beta(y) \hat{\psi}^{(+)}_\beta(y): \, |i\rangle$$
$$= \langle f| \, \hat{\bar{\psi}}^{(-)}_\beta(y) \hat{\bar{\psi}}^{(-)}_\alpha(x) \, \hat{\psi}^{(+)}_\alpha(x) \hat{\psi}^{(+)}_\beta(y) \, |i\rangle \,. \tag{12.55}$$

At this point we plug in the mode expansions of $\hat{\Psi}^{(+)}$ and $\hat{\bar{\Psi}}^{(-)}$, and calculate, first,

$$\hat{\psi}^{(+)}_\alpha(x) \hat{\psi}^{(+)}_\beta(y) \, |i\rangle = \sum_{s,s'} \int d^3p \, d^3p' \frac{\sqrt{E_A E_B}}{\sqrt{\omega_{\boldsymbol{p}} \omega_{\boldsymbol{p}'}}} u_\alpha(\boldsymbol{p}, s) e^{-ip\cdot x} u_\beta(\boldsymbol{p}', s') e^{-ip'\cdot y} \hat{b}_{\boldsymbol{p},s} \hat{b}_{\boldsymbol{p}',s'} \hat{b}^\dagger_{\boldsymbol{p}_A, s_A} \hat{b}^\dagger_{\boldsymbol{p}_B, s_B} |0\rangle \,, \tag{12.56}$$

where the operators can be reduced with a help of the anticommutator Leibniz rule $[A, BC] = \{A, B\}C - B\{A, C\}$, and the anticommutation relations (8.13) and (8.14):

$$\hat{b}_{\boldsymbol{p},s} \big[\hat{b}_{\boldsymbol{p}',s'}, \hat{b}^\dagger_{\boldsymbol{p}_A, s_A} \hat{b}^\dagger_{\boldsymbol{p}_B, s_B} \big] \, |0\rangle = \hat{b}_{\boldsymbol{p},s} \Big(\delta_{s', s_A} \delta(\boldsymbol{p}' - \boldsymbol{p}_A) \, \hat{b}^\dagger_{\boldsymbol{p}_B, s_B} - \delta_{s', s_B} \delta(\boldsymbol{p}' - \boldsymbol{p}_B) \, \hat{b}^\dagger_{\boldsymbol{p}_A, s_A} \Big) \, |0\rangle$$
$$= \Big(\delta_{s', s_A} \delta(\boldsymbol{p}' - \boldsymbol{p}_A) \, \delta_{s, s_B} \delta(\boldsymbol{p} - \boldsymbol{p}_B) - (A \leftrightarrow B) \Big) \, |0\rangle \,. \tag{12.57}$$

This leads to

$$\hat{\psi}^{(+)}_\alpha(x) \hat{\psi}^{(+)}_\beta(y) \, |i\rangle = \Big(u_\alpha(\boldsymbol{p}_B, s_B) u_\beta(\boldsymbol{p}_A, s_A) e^{-ip_B\cdot x} e^{-ip_A\cdot y} - (A \leftrightarrow B) \Big) \, |0\rangle \,. \tag{12.58}$$

Similarly,

$$\begin{aligned}\langle f|\,\hat{\bar{\psi}}^{(-)}_{\beta}(y)\hat{\bar{\psi}}^{(-)}_{\alpha}(x) &= \sum_{s,s'}\int d^3p\,d^3p'\,\frac{\sqrt{E_1E_2}}{\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'}}}\bar{u}_\alpha(\boldsymbol{p},s)e^{ip\cdot x}\bar{u}_\beta(\boldsymbol{p}',s')e^{ip'\cdot y}\,\langle 0|\,\hat{b}_{\boldsymbol{p}_2,s_2}\hat{b}_{\boldsymbol{p}_1,s_1}\hat{b}^\dagger_{\boldsymbol{p}',s'}\hat{b}^\dagger_{\boldsymbol{p},s}\\ &= \langle 0|\left(\bar{u}_\alpha(\boldsymbol{p}_2,s_2)\bar{u}_\beta(\boldsymbol{p}_1,s_1)e^{ip_2\cdot x}e^{ip_1\cdot y} - (1\leftrightarrow 2)\right), \end{aligned} \tag{12.59}$$

so, in total,

$$\begin{aligned}\langle f|\,:\hat{\bar{\Psi}}(x)\hat{\Psi}(x)\hat{\bar{\Psi}}(y)\hat{\Psi}(y):\,|i\rangle &= \bar{u}(\boldsymbol{p}_1,s_1)u(\boldsymbol{p}_A,s_A)\bar{u}(\boldsymbol{p}_2,s_2)u(\boldsymbol{p}_B,s_B)e^{i(p_2-p_B)\cdot x}e^{i(p_1-p_A)\cdot y}\\ &\quad - (A\leftrightarrow B) - (1\leftrightarrow 2) + (A\leftrightarrow B, 1\leftrightarrow 2). \end{aligned} \tag{12.60}$$

Recalling the formula (9.19) for the scalar Feynman propagator (M denoting the mass of the scalar particle), the scattering amplitude reads

$$\begin{aligned} S_{fi} \approx \frac{(-ig)^2}{2!}\int d^4x\,d^4y\int\frac{d^4q}{(2\pi)^4}\,\frac{i\,e^{-iq\cdot(x-y)}}{q^2-M^2+i\varepsilon}\\ \times\left(\bar{u}(1)u(A)\,\bar{u}(2)u(B)\,e^{i(p_2-p_B)\cdot x}e^{i(p_1-p_A)\cdot y} - (A\leftrightarrow B) - (1\leftrightarrow 2) + (A\leftrightarrow B, 1\leftrightarrow 2)\right), \end{aligned} \tag{12.61}$$

where we have adopted a shorthand notation $\bar{u}(1) = \bar{u}(p_1, s_1)$, etc. Note that under the x and y integration, the second term equals the third, and the first term equals the fourth one. The integrals can be easily performed, yielding

$$\begin{aligned} S_{fi} &\approx (-ig)^2\,i\int\frac{d^4q}{(2\pi)^4}\left(\frac{(2\pi)^4\delta(p_2-p_B-q)(2\pi)^4\delta(p_1-p_A+q)}{q^2-M^2+i\varepsilon}\,\bar{u}(1)u(A)\,\bar{u}(2)u(B)\,-(1\leftrightarrow 2)\right)\\ &= (2\pi)^4\delta(p_1+p_2-p_A-p_B)\left((-ig)^2\,\bar{u}(1)u(A)\,\frac{i}{(p_1-p_A)^2-M^2+i\varepsilon}\bar{u}(2)u(B)\,-(1\leftrightarrow 2)\right). \end{aligned} \tag{12.62}$$

Finally, we observe that the $i\varepsilon$ can be omitted as the denominator is never zero. To see this, note that in the center-of-mass frame $E_A = E_B = E_1 = E_2$, and hence $(p_1 - p_A)^2 = -(\boldsymbol{p}_1 - \boldsymbol{p}_A)^2 < 0$. By Eq. (12.10) we identify the invariant matrix element

$$i\mathcal{M}_{fi} \approx (-ig)^2\,\bar{u}(1)u(A)\,\frac{i}{(p_1-p_A)^2-M^2}\bar{u}(2)u(B)\,-(1\leftrightarrow 2). \tag{12.63}$$

Exercise 46. *Yukawa potential.* Find a potential V such that

$$\int d^3y\,e^{-i\boldsymbol{q}\cdot\boldsymbol{y}}\,V(\boldsymbol{y}) = -\frac{g^2}{\boldsymbol{q}^2+M^2}. \tag{12.64}$$

Solution:

Applying a further integration $\int d^3q\,e^{i\boldsymbol{q}\cdot\boldsymbol{x}}(\ldots)$ gives

$$(2\pi)^3V(\boldsymbol{x}) = -g^2\int d^3q\,\frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x}}}{\boldsymbol{q}^2+M^2}, \tag{12.65}$$

and adopting spherical coordinates in the $\boldsymbol{q}$-space, (q, θ, φ),

$$V(\boldsymbol{x}) = -g^2\int\frac{d^3q}{(2\pi)^3}\,\frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x}}}{\boldsymbol{q}^2+M^2} = -\frac{g^2}{(2\pi)^2}\int_0^\infty dq\,q^2\int_0^\pi d\theta\,\sin\theta\,\frac{e^{iqr\cos\theta}}{q^2+M^2}, \tag{12.66}$$

where $r \equiv |\boldsymbol{x}|$. Integration over the angle θ yields

$$V(\boldsymbol{x}) = -\frac{g^2}{(2\pi)^2 ir}\int_0^\infty dq\,\frac{q(e^{iqr}-e^{-iqr})}{q^2+M^2} = -\frac{g^2}{(2\pi)^2 ir}\int_{-\infty}^\infty dq\,\frac{q\,e^{iqr}}{q^2+M^2}. \tag{12.67}$$

This integral can be evaluated with a help of the Cauchy formula (9.58), taking as contours Γ_R the counterclockwise semicircles of radius R closed in the upper half of the complex q-plane (so that the integral over the arc vanishes in large-R limit). One finds

$$V(\boldsymbol{x}) = -\frac{g^2}{2\pi r}\oint_{\Gamma_\infty}\frac{dq}{2\pi i}\,\frac{q\,e^{iqr}}{(q-iM)(q+iM)} = -\frac{g^2}{4\pi}\frac{e^{-Mr}}{r}. \tag{12.68}$$

Chapter 13

Change of observer in quantum field theory

13.1 Poincaré transformations

In Chapter 2 we got acquainted with Lorentz transformations. Adding arbitrary (rigid) translations results in the full group of spacetime isometries (i.e., transformations preserving spacetime distances) — the *Poincaré* group. This has 10 parameters in total: 4 translations, 3 rotations, and 3 boosts. For concreteness, we will study the Poincaré transformations via their action on a generic multicomponent classical field $\Phi(x)$. (Here, unlike in the case of multicomponent scalar field in Section (7.2), the components of Φ can, and will, transform nontrivially under Lorentz transformations.)

For a translation $x' = x + a$, the induced transformation of Φ is

$$\Phi'(x') = \Phi(x) \quad \to \quad \Phi'(x) = \Phi(x-a) \approx \Phi(x) - ia^\mu(-i\partial_\mu)\Phi(x), \tag{13.1}$$

where the infinitesimal form allows us to identify the translation generators

$$\mathcal{P}_\mu = -i\partial_\mu. \tag{13.2}$$

For a Lorentz transformation $x' = \mathsf{L}x$, where $\mathsf{L} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathsf{M}^{\mu\nu}\right)$, we have $x'^\mu \approx x^\mu + \omega^\mu{}_\nu x^\nu$ on an infinitesimal level (see Eq. (2.19)), and the induced transformation on Φ reads (cf. Eq. 5.69)

$$\Phi'(x') = D(L)\Phi(x) \quad \to \quad \Phi'(x) = D(L)\Phi(\mathsf{L}^{-1}x), \quad \text{where} \quad D(L) = \exp\Big(-\frac{i}{2}\omega_{\mu\nu}\mathsf{S}^{\mu\nu}\Big) \tag{13.3}$$

is certain matrix representation of the Lorentz group generated by the matrices $\mathsf{S}^{\mu\nu}$. For example, for spinor fields, $D(L) = S(L)$ with $\mathsf{S}^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}$ (see Section 2.3). For infinitesimal parameters $\omega_{\mu\nu}$ we find

$$\begin{aligned}\Phi'(x) &\approx \Big(\mathbb{I} - \frac{i}{2}\omega_{\mu\nu}\mathsf{S}^{\mu\nu}\Big)\Phi(x^\mu - \omega^\mu{}_\nu x^\nu) \\ &\approx \Phi(x) - \frac{i}{2}\omega_{\mu\nu}\mathsf{S}^{\mu\nu}\Phi(x) - \omega^\mu{}_\nu x^\nu\partial_\mu\Phi(x) \\ &= \Phi(x) - \frac{i}{2}\omega_{\mu\nu}\big(\mathsf{S}^{\mu\nu} + i(x^\mu\partial^\nu - x^\nu\partial^\mu)\big)\Phi(x),\end{aligned} \tag{13.4}$$

hence identifying the Lorentz generators, which are composed of the orbital part + the internal spin part,

$$\mathcal{M}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \mathsf{S}_{\mu\nu}. \tag{13.5}$$

In summary, in classical field theory we encounter three different representations of Lorentz transformations: (1) the defining representation L on spacetime points, with generators $\mathsf{M}^{\mu\nu}$; (2) the internal representation $D(L)$ on components of the field, with generators $\mathsf{S}^{\mu\nu}$; and (3) the induced representation on functions $\Phi(x)$, with generators $\mathcal{M}^{\mu\nu}$.

Under a (continuous) symmetry transformation the quantum states are transformed by an operator that is unitary (in order to preserve the Fock space scalar product). For example, translations $x' = x + a$ are realized on the Fock space as

$$|\alpha'\rangle = \hat{U}(a)\,|\alpha\rangle\,. \tag{13.6}$$

Demanding that matrix elements of the quantum field operator $\hat{\Phi}(x)$ (such as, e.g., the one-particle wave-functions in Eqs. (7.60) and (8.29)) transform as classical fields, we find

$$\langle\beta'|\,\hat{\Phi}(x')\,|\alpha'\rangle = \langle\beta|\,\hat{\Phi}(x)\,|\alpha\rangle \quad \rightarrow \quad \hat{\Phi}(x+a) = \hat{U}(a)\hat{\Phi}(x)\hat{U}^\dagger(a) \quad \rightarrow \quad \hat{U}(a) = e^{ia^\mu \hat{P}_\mu}, \tag{13.7}$$

where we identified $\hat{U}(a)$ based on Eqs. (8.25) and (7.52).

For Lorentz transformations we recall Eq. (13.3). From

$$|\alpha'\rangle = \hat{U}(L)\,|\alpha\rangle \tag{13.8}$$

follows

$$\langle\beta'|\,\hat{\phi}_r(x')\,|\alpha'\rangle = D(L)_{rs}\,\langle\beta|\,\hat{\phi}_s(x)\,|\alpha\rangle \quad \rightarrow \quad D(L)^{-1}\hat{\Phi}(\mathsf{L}x) = \hat{U}(L)\hat{\Phi}(x)\hat{U}^\dagger(L). \tag{13.9}$$

The explicit form of the unitary operator $\hat{U}(L)$ is found with a help of the angular momentum tensor. By Section 5.4.2, Eq. (5.71) we have

$$M^{0\mu\nu} = -i\pi_r(\mathsf{S}^{\mu\nu})_{rs}\phi_s - (T^{0\mu}x^\nu - T^{0\nu}x^\mu) \quad \rightarrow \quad \hat{M}^{\mu\nu} = \int d^3x \;\, :\hat{M}^{0\mu\nu}: \,. \tag{13.10}$$

These are (6 independent) components of the total angular momentum operator. In fact, one may say total total angular momentum operator, as it is the angular momentum tensor integrated over space, which has both intrinsic (spin) and orbital part. In Exercise 47 we find the commutator

$$[\hat{\Phi}, \hat{M}^{\mu\nu}] = i(x^\mu\partial^\nu - x^\nu\partial^\mu)\hat{\Phi}(x) + \mathsf{S}^{\mu\nu}\hat{\Phi}(x) = \mathcal{M}^{\mu\nu}\hat{\Phi}(x), \tag{13.11}$$

which can be exponentiated by means of the Campbell identity (2.14), and comparison with Eq. (13.4) yields

$$e^{-\frac{i}{2}\omega_{\mu\nu}\hat{M}^{\mu\nu}}\,\hat{\Phi}(x)e^{\frac{i}{2}\omega_{\mu\nu}\hat{M}^{\mu\nu}} = e^{\frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}}\,\hat{\Phi}(x) = \lim_{\varepsilon\to 0}\Big(1+\varepsilon\frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}\Big)^{1/\varepsilon}\hat{\Phi}(x) = D(L)^{-1}\hat{\Phi}(\mathsf{L}x). \tag{13.12}$$

Thus, we may identify

$$\hat{U}(L) = e^{-\frac{i}{2}\omega_{\mu\nu}\hat{M}^{\mu\nu}}. \tag{13.13}$$

Note that since the operators $\hat{P}_\mu$ and $\hat{M}^{\mu\nu}$ are normal-ordered, the vacuum state is invariant under both translations and Lorentz transformations:

$$\hat{U}(a)\,|0\rangle = \hat{U}(L)\,|0\rangle = |0\rangle \tag{13.14}$$

This is consistent with the comment under Equation (6.20) — a symmetry of the system entails symmetry of its vacuum state. All observers that move with constant velocity (they are Lorentz-boosted with respect to one another) observe the same vacuum state. For mutually *accelerated* observers, however, the corresponding vacua differ.

13.2 Unruh effect

Unruh discovered that an observer accelerating in the Minkowski vacuum sees particles which have a thermal spectrum, with the temperature being proportional to the acceleration. This effect is called the Unruh effect, and in this chapter it will be derived in a simplified case. We shall consider a massless scalar field and assume that the observer is moving with constant acceleration a in 1+1-dimensional spacetime.

In [13, Ch. 8] it is argued the relation between the Minkowski inertial coordinates (t, x) and the (so-called Rindler) coordinates pertaining to the accelerated observer, namely τ (the proper time of the observer) and ξ (a convenient observer's distance measure), reads

$$t(\tau,\xi) = \frac{1}{a} e^{a\xi} \sinh a\tau \quad , \quad x(\tau,\xi) = \frac{1}{a} e^{a\xi} \cosh a\tau. \tag{13.15}$$

Note that for $\xi = 0$ this give the observer's trajectory and the four-acceleration w^μ

$$x^\mu_{\text{obs}} = (t(\tau,0), x(\tau,0)) \quad \to \quad w^\mu = \frac{d^2 x^\mu_{\text{obs}}}{d\tau^2} = a\,(\sinh a\tau, \cosh a\tau) \quad \to \quad w^\mu w_\mu = -a^2. \tag{13.16}$$

To analyze the equation of motion of a massless scalar field in one spatial dimension, i.e., the standard wave equation, it is convenient to introduce the 'light-cone' coordinates (as in [9, Ch. 2.8])

$$u = t - x \quad , \quad v = t + x : \qquad (\partial_t^2 - \partial_x^2)\phi = 0 \quad \to \quad \partial_u \partial_v \phi = 0. \tag{13.17}$$

Casting

$$\begin{aligned} u &= -\frac{1}{a} e^{a\xi} e^{-a\tau} = -\frac{1}{a} e^{-a\tilde{u}}, \quad \text{where} \quad \tilde{u} = \tau - \xi, \\ v &= \frac{1}{a} e^{a\xi} e^{a\tau} = \frac{1}{a} e^{a\tilde{v}}, \quad \text{where} \quad \tilde{v} = \tau + \xi \end{aligned} \tag{13.18}$$

identifies $\tilde{u}$ and $\tilde{v}$ as the light-cone coordinates corresponding to τ, ξ. Note that, crucially, u is a function of $\tilde{u}$ only, and likewise v is a function of $\tilde{v}$ only. Hence

$$\partial_u \partial_v \phi = \frac{d\tilde{u}}{du} \frac{d\tilde{v}}{dv}\, \partial_{\tilde{u}} \partial_{\tilde{v}} \phi = 0 \quad \to \quad (\partial_\tau^2 - \partial_\xi^2)\phi = 0. \tag{13.19}$$

We have found that the field equation has the same form in both the inertial and the accelerated frame.

In both frames we may immediately write a general solution in the form of a mode expansion of the quantized field $\hat{\phi}$. The massless dispersion relation in one-dimensional space reads $\omega_p = |p|$, where $p \in \mathbb{R}$, and hence, in Minkowski coordinates,

$$\begin{aligned} \hat{\phi} &= \int_{-\infty}^{+\infty} \frac{dp}{\sqrt{2\pi\, 2|p|}} \Big(\hat{a}_p\, e^{-i(|p|t - px)} + \hat{a}_p^\dagger\, e^{i(|p|t - px)} \Big) \\ &= \int_0^\infty \frac{d\omega}{\sqrt{2\pi\, 2\omega}} \Big(\hat{a}_\omega\, e^{-i\omega u} + \hat{a}_\omega^\dagger\, e^{i\omega u} + \hat{a}_{-\omega}\, e^{-i\omega v} + \hat{a}_{-\omega}^\dagger\, e^{i\omega v} \Big), \end{aligned} \tag{13.20}$$

where we have separated the right-moving modes (with $p = \omega > 0$) and the left-moving modes (with $p = -\omega < 0$). In the Rindler coordinates, likewise

$$\begin{aligned} \hat{\phi} &= \int_{-\infty}^{+\infty} \frac{d\tilde{p}}{\sqrt{2\pi\, 2|\tilde{p}|}} \Big(\hat{b}_{\tilde{p}}\, e^{-i(|\tilde{p}|\tau - \tilde{p}\xi)} + \hat{b}_{\tilde{p}}^\dagger\, e^{i(|\tilde{p}|\tau - \tilde{p}\xi)} \Big) \\ &= \int_0^\infty \frac{d\Omega}{\sqrt{2\pi\, 2\Omega}} \Big(\hat{b}_\Omega\, e^{-i\Omega\tilde{u}} + \hat{b}_\Omega^\dagger\, e^{i\Omega\tilde{u}} + \hat{b}_{-\Omega}\, e^{-i\Omega\tilde{v}} + \hat{b}_{-\Omega}^\dagger\, e^{i\Omega\tilde{v}} \Big). \end{aligned} \tag{13.21}$$

The u and v parts can be considered separately, and so will focus only on the u part (the v part is treated in complete analogy). Equating the two expansions (13.20) and (13.21),

$$\int_0^\infty \frac{d\Omega}{\sqrt{2\pi\, 2\Omega}}\Big(\hat b_\Omega\, e^{-i\Omega\tilde u}+\hat b_\Omega^\dagger\, e^{i\Omega\tilde u}\Big)=\int_0^\infty \frac{d\omega}{\sqrt{2\pi\, 2\omega}}\Big(\hat a_\omega\, e^{-i\omega u}+\hat a_\omega^\dagger\, e^{i\omega u}\Big),\quad \text{where}\quad u=-\frac{1}{a}e^{-a\tilde u}. \tag{13.22}$$

provides a relation between the inertial observer's creation and annihilation operators $\hat a^\dagger, \hat a$, and the accelerated observer's creation and annihilation operators $\hat b^\dagger, \hat b$. We shall see that while the Minkowski vacuum state $|0\rangle$ is annihilate by all $\hat a_p$, it is not annihilated by the operators $\hat b_{\tilde p}$ (and therefore contains particles from the point of view of the accelerated observer).

We would like to have an explicit expression for $\hat b_\Omega$ in terms of $\hat a$ and $\hat a^\dagger$. To this end let us apply the integral $\int_{-\infty}^{+\infty}\frac{d\tilde u}{\sqrt{2\pi}}\, e^{i\Omega'\tilde u}(\ldots)$, with $\Omega'>0$, on the left-hand side of Eq. (13.22):

$$\int_0^\infty \frac{d\Omega}{\sqrt{2\Omega}}\Big(\hat b_\Omega\, \delta(\Omega'-\Omega)+\hat b_\Omega^\dagger\, \delta(\Omega'+\Omega)\Big)=\frac{\hat b_{\Omega'}}{\sqrt{2\Omega'}}, \tag{13.23}$$

since the argument of $\delta(\Omega'+\Omega)$ is always greater than 0. On the right-hand side we get

$$\int_0^\infty \frac{d\omega}{\sqrt{2\omega}}\Big(F(\omega,\Omega')\,\hat a_\omega+F(-\omega,\Omega')\,\hat a_\omega^\dagger\Big),\quad \text{where}\quad F(\omega,\Omega)=\int_{-\infty}^{+\infty}\frac{d\tilde u}{2\pi}\exp\Big(i\Omega\tilde u+i\frac{\omega}{a}e^{-a\tilde u}\Big). \tag{13.24}$$

Therefore,

$$\hat b_\Omega=\int_0^\infty d\omega\sqrt{\frac{\Omega}{\omega}}\Big(F(\omega,\Omega)\,\hat a_\omega+F(-\omega,\Omega)\,\hat a_\omega^\dagger\Big), \tag{13.25}$$

and, by Hermitian conjugation,

$$\hat b_\Omega^\dagger=\int_0^\infty d\omega\sqrt{\frac{\Omega}{\omega}}\Big(F^*(-\omega,\Omega)\,\hat a_\omega+F^*(\omega,\Omega)\,\hat a_\omega^\dagger\Big). \tag{13.26}$$

The mean number of particles with momentum Ω that the accelerated observer registers in the Minkowski vacuum $|0\rangle$ reads

$$\langle 0|\,\hat b_\Omega^\dagger \hat b_\Omega\,|0\rangle=\int_0^\infty d\omega\, d\omega'\,\frac{\Omega}{\sqrt{\omega\omega'}}\,F^*(-\omega,\Omega)F(-\omega',\Omega)\underbrace{\langle 0|\,\hat a_\omega \hat a_{\omega'}^\dagger\,|0\rangle}_{\delta(\omega-\omega')}=\int_0^\infty d\omega\,\frac{\Omega}{\omega}\,|F(-\omega,\Omega)|^2. \tag{13.27}$$

The last integral can be inferred even without an explicit formula for the function F. First, from the commutation relations

$$[\hat a_\omega,\hat a_{\omega'}^\dagger]=\delta(\omega-\omega')\quad(\text{otherwise }0)\qquad\text{and}\qquad[\hat b_\Omega,\hat b_{\Omega'}^\dagger]=\delta(\Omega-\Omega')\quad(\text{otherwise }0) \tag{13.28}$$

follows

$$\begin{aligned}[\hat b_\Omega,\hat b_\Omega^\dagger]&=\int_0^\infty d\omega\, d\omega'\frac{\Omega}{\sqrt{\omega\omega'}}\Big(F(\omega,\Omega)F^*(\omega',\Omega)\underbrace{[\hat a_\omega,\hat a_{\omega'}^\dagger]}_{\delta(\omega-\omega')}+F(-\omega,\Omega)F^*(-\omega',\Omega)\underbrace{[\hat a_\omega^\dagger,\hat a_{\omega'}]}_{-\delta(\omega-\omega')}\Big)\\&=\int_0^\infty d\omega\,\frac{\Omega}{\omega}\Big(|F(\omega,\Omega)|^2-|F(-\omega,\Omega)|^2\Big)\end{aligned} \tag{13.29}$$

At the same time,

$$[\hat b_\Omega,\hat b_\Omega^\dagger]=\delta(\Omega-\Omega)=\delta(0)=\frac{V}{2\pi}, \tag{13.30}$$

where V is the (one dimensional) volume of space. Second, substituting $\tilde{u} \to \tilde{u} - i\frac{\pi}{a}$ (or rather moving the integration contour) in the integral definition of F, Eq. (13.24), we derive the relation

$$F(\omega, \Omega) = \int_{-\infty}^{+\infty} \frac{d\tilde{u}}{2\pi} \exp\left(i\Omega\tilde{u} + \frac{\Omega\pi}{a} - i\frac{\omega}{a}e^{-a\tilde{u}}\right) = F(-\omega, \Omega)\, e^{\pi\Omega/a}. \tag{13.31}$$

Comparing Eqs. (13.29) and (13.30), and making use of Eq. (13.31),

$$\frac{V}{2\pi} = \left(e^{2\pi\Omega/a} - 1\right) \int_0^{\infty} d\omega\, \frac{\Omega}{\omega}\, |F(-\omega, \Omega)|^2. \tag{13.32}$$

This can be used in Eq. (13.27) to finally find the mean density of particles in a mode with momentum $p = \Omega$ (or $p = -\Omega$ for the left-moving particles),

$$n(\Omega) = \frac{1}{V} \langle 0|\, \hat{b}_\Omega^\dagger \hat{b}_\Omega \,|0\rangle = \frac{1}{2\pi} \frac{1}{e^{2\pi\Omega/a} - 1}. \tag{13.33}$$

This has a form of Bose-Einstein distribution ($E = \hbar\Omega$):

$$n(E) \propto \frac{1}{e^{2\pi E/a} - 1} \quad \to \quad \frac{2\pi}{a} = \frac{1}{k_B T}, \quad \text{where} \quad \frac{T}{a} = \frac{\hbar}{2\pi c k_B}. \tag{13.34}$$

T is the Unruh temperature. An accelerated detector perceives a thermal bath with temperature T.

13.3 Exercises

Exercise 47. *Angular momentum field operator.* Consider a multicomponent (bosonic or fermionic) field $\Phi = (\phi_r)$, and show that the *total angular momentum* operator

$$\hat{M}^{\mu\nu} = \int d^3x \ :\hat{M}^{0\mu\nu}:, \quad \text{where} \quad M^{0\mu\nu} = -i\pi_r(\mathsf{S}^{\mu\nu})_{rs}\phi_s - (T^{0\mu}x^\nu - T^{0\nu}x^\mu), \tag{13.35}$$

enjoys the commutation property

$$[\hat{\Phi}(x), \hat{M}^{\mu\nu}] = \big(i(x^\mu\partial^\nu - x^\nu\partial^\mu) + \mathsf{S}^{\mu\nu}\big)\hat{\Phi}(x). \tag{13.36}$$

Solution:

When calculating the commutator we may omit the normal ordering, since the operators $\hat{M}^{0\mu\nu}$ and $:\hat{M}^{0\mu\nu}:$ differ only by a constant number. In the following we set $y^0 = x^0$ ($\hat{M}^{\mu\nu}$ is constant in time):

$$\begin{aligned}[\hat{\phi}_r(x), \hat{M}^{\mu\nu}] &= \int d^3y \Big(- i(\mathsf{S}^{\mu\nu})_{r's}\big[\hat{\phi}_r(x), \hat{\pi}_{r'}(y)\hat{\phi}_s(y)\big] - \big[\hat{\phi}_r(x), \hat{T}^{0\mu}(y)y^\nu - \hat{T}^{0\nu}(y)y^\mu\big]\Big) \\ &= (\mathsf{S}^{\mu\nu})_{rs}\hat{\phi}_s(x) - \int d^3y \Big(\big[\hat{\phi}_r(x), \hat{T}^{0\mu}(y)\big]y^\nu - \big[\hat{\phi}_r(x), \hat{T}^{0\nu}(y)\big]y^\mu\Big)\end{aligned} \tag{13.37}$$

Here we have used canonical commutation (bosonic) or anticommutation (fermionic) relations

$$[\hat{\phi}_r(\boldsymbol{x},t), \hat{\pi}_s(\boldsymbol{y},t)] = i\,\delta_{rs}\,\delta(\boldsymbol{x}-\boldsymbol{y}) \quad \text{or} \quad \{\hat{\phi}_r(\boldsymbol{x},t), \hat{\pi}_s(\boldsymbol{y},t)\} = i\,\delta_{rs}\,\delta(\boldsymbol{x}-\boldsymbol{y}) \quad \text{(and zero otherwise)}, \tag{13.38}$$

and Leibniz rules $[AB, C] = A[B, C] + [A, C]B$ or $[AB, C] = A\{B, C\} - \{A, C\}B$, respectively.

Now recall that $T^{0\mu} = \pi_r\partial^\mu\phi_r - g^{0\mu}\mathscr{L}$, and hence

$$[\hat{\phi}_r(x), \hat{T}^{0i}(y)] = [\hat{\phi}_r(x), \hat{\pi}_s(y)\partial^i\hat{\phi}_s(y)] = i\delta_{rs}\delta(\boldsymbol{x}-\boldsymbol{y})\partial^i\hat{\phi}_s(y) = i\delta(\boldsymbol{x}-\boldsymbol{y})\partial^i\hat{\phi}_r(x). \tag{13.39}$$

Moreover, since $T^{00} = \mathscr{H}$ and $H = \int d^3y\,\mathscr{H}(y)$, considering the Heisenberg equation

$$i\partial_0\hat{\phi}_r(x) = [\hat{\phi}_r(x), \hat{H}] \quad \text{yields} \quad [\hat{\phi}_r(x), \hat{T}^{00}(y)] = [\hat{\phi}_r(x), \hat{\mathscr{H}}(y)] = i\delta(\boldsymbol{x}-\boldsymbol{y})\partial_0\hat{\phi}_r(x). \tag{13.40}$$

Thus, we have

$$[\hat{\phi}_r(x), \hat{T}^{0\mu}(y)] = i\delta(\boldsymbol{x}-\boldsymbol{y})\partial^\mu\hat{\phi}_r(x), \tag{13.41}$$

and may finally calculate

$$\begin{aligned}[\hat{\phi}_r(x), \hat{M}^{\mu\nu}] &= (\mathsf{S}^{\mu\nu})_{rs}\hat{\phi}_s(x) - \int d^3y \Big(y^\nu i\delta(\boldsymbol{x}-\boldsymbol{y})\partial^\mu\hat{\phi}_r(x) - y^\mu i\delta(\boldsymbol{x}-\boldsymbol{y})\partial^\nu\hat{\phi}_r(x)\Big) \\ &= (\mathsf{S}^{\mu\nu})_{rs}\hat{\phi}_s(x) + i(x^\mu\partial^\nu - x^\nu\partial^\mu)\hat{\phi}_r(x).\end{aligned} \tag{13.42}$$

Appendix A

Exam questions

1. Relativistic wave equations: Klein-Gordon equation, First-order equations (Dirac's method)
2. Lorentz group and Lorentz algebra, Clifford algebra, Representations of gamma matrices
3. Dirac equation and its Lorentz covariance, Plane wave solutions, Dirac current and other bilinears
4. Dirac particle in electromagnetic field, Non-relativistic limit of Dirac equation, Energy levels of a Dirac particle in Coulomb potential
5. Discrete transformations of Dirac equation (Charge conjugation, Parity, Time reversal), Helicity and chirality
6. Classical systems of coupled oscillators, Normal modes, Quantum systems of coupled oscillators, Vacuum state
7. Functional derivatives, Lagrangian and Hamiltonian formalism in classical field theory
8. Symmetries and conservation laws (Noether theorem) in field theory: Translations, Lorentz transformations, Internal rotations
9. Field theory for quantum non-relativistic many-particle systems: Bosonic systems, Fermionic systems, Interparticle interactions
10. Canonical quantisation of Klein-Gordon field, Mode expansion, Total four-momentum, Normal ordering
11. Multiplet of scalar fields, Charged Klein-Gordon field, States and particle interpretation
12. Canonical quantisation of Dirac field, Mode expansion, States and conserved quantities, Spin-statistics connection
13. Pauli-Jordan commutation function, Feynman propagator for Klein-Gordon field, Retarded (Forward) propagator
14. Inferring propagators from the action, Feynman propagator for Dirac field, Feynman propagator for Schrödinger field
15. Classical electromagnetism, Covariant canonical quantization of electromagnetic field, Proca field

16. Interacting quantum fields, Interaction picture, Dyson operator, Scattering matrix
17. Wick theorem in generating form, Wick expansion of time-ordered products, Vacuum expectation values, Diagrammatic representation
18. Applications of quantum field theory in particle physics: Decay of an unstable particle, Scattering cross section
19. Change of observer in quantum field theory: Poincaré transformations and Unruh effect

Appendix B

Selected formulas

B.1 Relativistic quantum mechanics

Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \sigma^i\sigma^j = \delta_{ij} + i\,\varepsilon_{ijk}\sigma^k \tag{B.1}$$

Baker-Campbell-Hausdorff formula (restricted):

$$e^{\mathsf{A}}e^{\mathsf{B}} = e^{\mathsf{A}+\mathsf{B}}e^{\frac{1}{2}[\mathsf{A},\mathsf{B}]} \qquad \text{provided} \qquad [\mathsf{A},[\mathsf{A},\mathsf{B}]] = [\mathsf{B},[\mathsf{A},\mathsf{B}]] = 0 \tag{B.2}$$

Campbell identity:

$$e^{\mathsf{A}}\,\mathsf{B}\,e^{-\mathsf{A}} = \mathsf{B} + [\mathsf{A},\mathsf{B}] + \frac{1}{2!}[\mathsf{A},[\mathsf{A},\mathsf{B}]] + \ldots = \sum_{n=0}^{\infty}\frac{\mathsf{K}_n}{n!}, \quad \text{where} \quad \mathsf{K}_0 = \mathsf{B}, \quad \mathsf{K}_{n+1} = [\mathsf{A},\mathsf{K}_n] \quad (\forall n) \tag{B.3}$$

Lorentz generators:

$$(\mathsf{M}^{\mu\nu})^{\rho}{}_{\sigma} = i(g^{\mu\rho}\delta^{\nu}_{\sigma} - g^{\nu\rho}\delta^{\mu}_{\sigma}) \quad , \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu},\gamma^{\nu}] \tag{B.4}$$

Finite Lorentz transformations:

$$\mathsf{L} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathsf{M}^{\mu\nu}\right) \quad , \quad S(L) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) \quad , \quad S(L)^{-1}\gamma^{\mu}S(L) = L^{\mu}{}_{\nu}\gamma^{\nu} \tag{B.5}$$

Rotations and boosts:

$$\mathsf{J}_i \equiv \frac{1}{2}\varepsilon_{ijk}\mathsf{M}^{jk}, \quad \mathsf{K}_i \equiv \mathsf{M}^{0i} \quad , \quad \omega_{ij} = \varepsilon_{ijk}\theta^k, \quad \omega_{0i} = \zeta^i : \quad \mathsf{L} = \exp\left(-i\theta^i\mathsf{J}_i - i\zeta^i\mathsf{K}_i\right) \tag{B.6}$$

Rotations and boosts (spin representation):

$$S(L)_{D,W} = \exp\left(-\frac{i}{2}\theta^i\Sigma^i - \frac{i}{2}\zeta^i\sigma^{0i}_{D,W}\right), \quad \Sigma^i = \mathbb{I}\otimes\sigma^i, \quad \sigma^{0i}_D = i\begin{pmatrix} \mathbb{O} & \sigma^i \\ \sigma^i & \mathbb{O} \end{pmatrix}, \ \sigma^{0i}_W = i\begin{pmatrix} -\sigma^i & \mathbb{O} \\ \mathbb{O} & \sigma^i \end{pmatrix} \tag{B.7}$$

Dirac and Weyl representation:

$$\gamma^0_D = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix} = -\gamma^5_W \quad , \quad \gamma^i_D = \begin{pmatrix} \mathbb{O} & \sigma^i \\ -\sigma^i & \mathbb{O} \end{pmatrix} = \gamma^i_W \quad , \quad \gamma^5_D = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix} = \gamma^0_W \tag{B.8}$$

Spin sums:

$$\sum_s u(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s)=\frac{\gamma^\mu p_\mu+m}{2m}\quad,\quad\sum_s v(\boldsymbol{p},s)\bar{v}(\boldsymbol{p},s)=\frac{\gamma^\mu p_\mu-m}{2m}\tag{B.9}$$

Orthogonality of polarization spinors I:

$$\bar{u}(\boldsymbol{p},s)u(\boldsymbol{p},s')=\delta_{ss'}\quad,\quad\bar{v}(\boldsymbol{p},s)v(\boldsymbol{p},s')=-\delta_{ss'}\quad,\quad\bar{u}(\boldsymbol{p},s)v(\boldsymbol{p},s')=0\tag{B.10}$$

Orthogonality of polarization spinors II:

$$u^\dagger(\boldsymbol{p},s)u(\boldsymbol{p},s')=\frac{\omega_{\boldsymbol{p}}}{m}\delta_{ss'}\quad,\quad v^\dagger(\boldsymbol{p},s)v(\boldsymbol{p},s')=\frac{\omega_{\boldsymbol{p}}}{m}\delta_{ss'}\quad,\quad u^\dagger(\boldsymbol{p},s)v(-\boldsymbol{p},s')=0\tag{B.11}$$

Discrete transformations:

$$\Psi_C(x)=\gamma^2\Psi^*(x)\quad,\quad\Psi_P(x^0,-\boldsymbol{x})=\gamma^0\Psi(x)\quad,\quad\Psi_T(-x^0,\boldsymbol{x})=\gamma^1\gamma^3\Psi^*(x)\tag{B.12}$$

B.2 Quantum field theory

Noether theorem:

$$x'^\mu=x^\mu+\delta x^\mu(x)\ ,\ \phi'_r(x')=\phi_r(x)+\delta\phi_r(x)\ ,\ f^\mu=\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}\delta\phi_r-\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}\partial_\nu\phi_r-\delta^\mu_\nu\mathscr{L}\right)\delta x^\nu\tag{B.13}$$

Energy-momentum tensor:

$$T^\mu{}_\nu=\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}\partial_\nu\phi_r-\delta^\mu_\nu\mathscr{L}\tag{B.14}$$

Angular momentum tensor:

$$M^{\mu\nu\rho}=-i\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_r)}(\mathsf{S}^{\nu\rho})_{rs}\phi_s-(T^{\mu\nu}x^\rho-T^{\mu\rho}x^\nu)\tag{B.15}$$

Schrödinger field:

$$\mathscr{L}=i\hbar\psi^*\partial_t\psi-\frac{\hbar^2}{2m}(\partial_i\psi^*)(\partial_i\psi)-V(\boldsymbol{x})\psi^*\psi\tag{B.16}$$

Real Klein-Gordon field:

$$\mathscr{L}=\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{1}{2}m^2\phi^2\quad,\quad\hat{\phi}(x)=\int\frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}}\left(\hat{a}_{\boldsymbol{p}}\,e^{-ip\cdot x}+\hat{a}^\dagger_{\boldsymbol{p}}\,e^{ip\cdot x}\right)\quad(\omega_{\boldsymbol{p}}\equiv\sqrt{\boldsymbol{p}^2+m^2})\tag{B.17}$$

Complex Klein-Gordon field:

$$\mathscr{L}=(\partial_\mu\varphi^*)(\partial^\mu\varphi)-m^2\varphi^*\varphi\quad,\quad\hat{\varphi}(x)=\int\frac{d^3p}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{p}}}}\left(\hat{a}_{\boldsymbol{p}}\,e^{-ip\cdot x}+\hat{b}^\dagger_{\boldsymbol{p}}\,e^{ip\cdot x}\right)\tag{B.18}$$

Dirac field:

$$\mathscr{L}=\bar{\Psi}(i\gamma^\mu\partial_\mu-m)\Psi\quad,\quad\hat{\Psi}(x)=\sum_{s=\pm\frac{1}{2}}\int\frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{\omega_{\boldsymbol{p}}}}\left(\hat{b}_{\boldsymbol{p},s}u(\boldsymbol{p},s)e^{-ip\cdot x}+\hat{d}^\dagger_{\boldsymbol{p},s}v(\boldsymbol{p},s)e^{ip\cdot x}\right)\tag{B.19}$$

— polarization spinors:

$$u(\boldsymbol{p},s)=\frac{\gamma^\mu p_\mu+m}{\sqrt{2m(p_0+m)}}\,u(\boldsymbol{0},s)\quad,\quad v(\boldsymbol{p},s)=\frac{-\gamma^\mu p_\mu+m}{\sqrt{2m(p_0+m)}}\,v(\boldsymbol{0},s),$$

$$u(\boldsymbol{0},s)=\begin{pmatrix}1\\0\end{pmatrix}\otimes\chi_s\quad,\quad v(\boldsymbol{0},s)=\begin{pmatrix}0\\1\end{pmatrix}\otimes\chi_s\quad,\quad \chi_{\frac12}=\begin{pmatrix}1\\0\end{pmatrix}\quad,\quad \chi_{-\frac12}=\begin{pmatrix}0\\1\end{pmatrix}\tag{B.20}$$

Electromagnetic field:

$$\mathscr{L}_\xi=-\frac14F_{\mu\nu}F^{\mu\nu}-\frac{1}{2\xi}(\partial_\rho A^\rho)^2$$

$$\hat{A}_\mu(x)=\sum_{\lambda=0}^{3}\int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{k}}}}\Big(\hat{a}_{\boldsymbol{k},\lambda}\,\varepsilon_\mu(\boldsymbol{k},\lambda)\,e^{-ik\cdot x}+\hat{a}^\dagger_{\boldsymbol{k},\lambda}\,\varepsilon_\mu(\boldsymbol{k},\lambda)\,e^{ik\cdot x}\Big)\tag{B.21}$$

— polarization vectors:

$$\varepsilon^\mu(\boldsymbol{k},0)=(1,\boldsymbol{0})\quad,\quad\varepsilon^\mu(\boldsymbol{k},i)=(0,\boldsymbol{\varepsilon}_i(\boldsymbol{k})),\quad,\quad\boldsymbol{\varepsilon}_3(\boldsymbol{k})=\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\quad,\quad\boldsymbol{\varepsilon}_i(\boldsymbol{k})\cdot\boldsymbol{\varepsilon}_j(\boldsymbol{k})=\delta_{ij}\tag{B.22}$$

Feynman propagators:

$$\begin{aligned}\text{Klein-Gordon}:&\quad\Delta_F(x-y)=\int\frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\varepsilon}\\ \text{Dirac}:&\quad S_F(x-y)=(i\partial\!\!\!/+m)\Delta_F(x-y)\\ \text{Electromagnetic}:&\quad(D_F)_{\mu\nu}(x-y)=-g_{\mu\nu}\,\Delta_F(x-y)|_{m=0}\end{aligned}\tag{B.23}$$

Scattering matrix:

$$S_{fi}=\langle f|\,\hat{U}(+\infty,-\infty)\,|i\rangle\quad,\quad\hat{U}(t,t_0)=T\exp\left(-i\int_{t_0}^{t}dt'\,\hat{H}_I^I(t')\right)\quad,\quad H_I=-\int d^3x\,\mathscr{L}_I\tag{B.24}$$

Wick theorem (for real one-component scalar field):

$$T\exp\Big(i\int d^4xJ(x)\hat{\phi}(x)\Big)=\,:\exp\Big(i\int d^4xJ(x)\hat{\phi}(x)\Big):\,\exp\Big(-\frac12\int d^4xd^4yJ(x)\,i\Delta_F(x-y)\,J(y)\Big)\tag{B.25}$$

Invariant matrix element:

$$S_{fi}=\delta_{fi}+(2\pi)^4\delta(p_f-p_i)\,i\mathcal{M}_{fi}\tag{B.26}$$

Particle states:

$$\begin{aligned}\text{spin-0}:&\quad|\boldsymbol{p}\rangle=\sqrt{(2\pi)^3 2E}\,\hat{a}^\dagger_{\boldsymbol{p}}\,|0\rangle\\ \text{spin-}\frac12:&\quad|\boldsymbol{p},s\rangle=\frac{\sqrt{(2\pi)^3 2E}}{\sqrt{2m}}\,\hat{b}^\dagger_{\boldsymbol{p},s}\,|0\rangle\end{aligned}\tag{B.27}$$

Decay rate $A\to1+2$ (rest frame, $m_1=m_2$):

$$\frac{d\Gamma}{d\Omega}=\frac{N_AN_1N_2}{64\pi^2m_A^2}\sqrt{m_A^2-4m_1^2}\,|\mathcal{M}_{fi}|^2\quad,\quad N=\begin{cases}1&\text{for bosons}\\2m&\text{for fermions}\end{cases}\tag{B.28}$$

Scattering cross section $A+B\to1+2$ (center-of-mass frame, $m_A=m_B=m_1=m_2$):

$$\frac{d\sigma}{d\Omega}=\frac{N_AN_BN_1N_2}{64\pi^2s}\,|\mathcal{M}_{fi}|^2\quad,\quad s=(p_A+p_B)^2\tag{B.29}$$

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