Real spinors and real Dirac equation

Václav Zatloukal

FNSPE, Czech Technical University in Prague

[www.zatlovac.eu]

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Motivation

$$i\hbar\partial_t\psi = \hat{H}\psi \quad , \quad (i\gamma^\mu\partial_\mu - m)\psi = 0 \quad , \quad \int \mathcal{D}\phi \exp\left(\frac{i}{\hbar}S[\phi, J]\right)$$

Find geometric meaning of i, the unit imaginary of quantum theory.

Make use of a rich geometric structure of real Clifford algebra (aka. *geometric algebra*).

Following:

D. Hestenes, *Spacetime Physics with Geometric Algebra*, Am. J. Phys. **71** 6, (2003)

C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, CUP (2007)

Outline

- Real Clifford algebras:
  rotations and real spinors in $2D$, $3D$ and $3+1D$
- Dirac equation:
  matrix form vs. real form, gauge fields

V. Z., *Real spinors and real Dirac equation*, arXiv:1908.04590 (2019)

Real Clifford algebra

Real vector space $V = span\{e_1, \dots, e_n\}$
with quadratic form $Q(e_j) = \pm 1$ of signature (p, q)

$\rightarrow$ *Real* Clifford algebra: "freest" associative algebra generated by V, subject to $a^2 = Q(a)$, $\forall a \in V$

$$\mathcal{C}\ell(V^{p,q}) = span\{1,\ e_i,\ e_i e_j (i<j),\ \dots,\ e_1 e_2 \dots e_n\} \quad (dim = 2^n)$$

Clifford (geometric) product: $ab = a \cdot b + a \wedge b \quad (a, b \in V)$

$a \cdot b = \frac{1}{2}(ab + ba)$ is a scalar (by $(a+b)^2 = a^2 + b^2 + ab + ba$)
$a \wedge b = \frac{1}{2}(ab - ba)$ is a bivector

Matrix representation: $e_1, \dots, e_n \leftrightarrow \gamma$-matrices

Geometric representation: r-vectors $\leftrightarrow$ r-dim. parallelograms
D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, Springer (1987).

Even subalgebra: $\mathcal{C}\ell_{even}(V^{p,q}) = span\{1,\ e_i e_j (i<j),\ \dots\} \quad (dim = 2^{n-1})$

2D: Rotations and complex numbers

Euclidean plane $E^2 = span\{e_1, e_2\}$

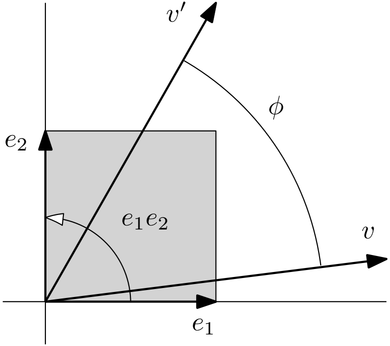
[$e_1^2 = e_2^2 = 1$, $e_1 \cdot e_2 = 0$]

$\rightarrow$ Clifford algebra

$\mathcal{C}\ell(E^2) = span\{1,\ e_1, e_2,\ e_1 e_2\}$

Even subalgebra

$\mathcal{C}\ell_{even}(E^2) = span\{1,\ e_1 e_2\}$



'Complex numbers': $x + y\, e_1 e_2 = r\, e^{\phi e_1 e_2}$ [$(e_1 e_2)^2 = -1$]

Rotation by angle ϕ:

$$v \ \mapsto\ v\, e^{\phi e_1 e_2} = e^{-\frac{\phi}{2} e_1 e_2} v\, e^{\frac{\phi}{2} e_1 e_2} \tag{1}$$

3D: Rotations and real spinors

Euclidean space $E^3 = span\{e_1, e_2, e_3\}$

$[\ e_i \cdot e_j = \delta_{ij}\ ]$

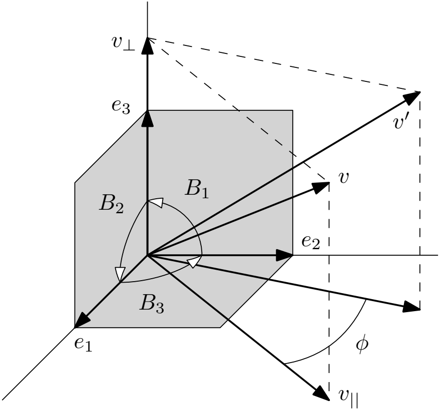
$\rightarrow$ Clifford algebra $\mathcal{C}\ell(E^3) =$

$span\{1,\ e_1, e_2, e_3,\ B_1, B_2, B_3,\ e_1 e_2 e_3\}$

$[\ B_i = \frac{\varepsilon_{ijk}}{2} e_j e_k \leftrightarrow$ quaternionic units]

Rotation in B-plane by angle $|B|$:

$$v \mapsto U\, v\, U^{-1} \quad , \quad U = e^{-B/2} \quad (2)$$



Rotors U form group $Spin(3)$ (double-cover of $SO(3)$)

Spinors: $\psi \in \mathcal{C}\ell_{even}(E^3) = span\{1,\ B_1, B_2, B_3\} = span\{e^{-B/2}\,|\,B \in \mathcal{C}\ell_2\}$

Spinorial representation: $U\psi$ (cf. $U\psi U^{-1} \Leftarrow v_1 \dots v_r \mapsto U v_1 \dots v_r U^{-1}$)

3D: Real spinors and Pauli spinors

Real spinor $\psi = \alpha_0 + \alpha_i B_i = \sqrt{\rho} R$ (rotor with magnitude)

(4 *real* components $\leftrightarrow$ 2 *complex* numbers)

$\rightarrow$ **Pauli spinor**

$$|\psi\rangle \equiv \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \in \mathbb{C}^2 \quad , \qquad \begin{array}{l} z_0 = \langle \psi \, (1 - i \, B_3) \rangle \\ z_1 = \langle B_2 \, \psi \, (1 - i \, B_3) \rangle \end{array} \tag{3}$$

Scalar part $\langle \ . \ \rangle$, $\langle AB \rangle = \langle BA \rangle$ (cf. $tr(\ . \ )$ in matrix representation)

Algebraic operations on $\psi \leftrightarrow$ Matrix operations on $|\psi\rangle$:

$$|B_j \psi\rangle = i\hat{\sigma}_j |\psi\rangle$$

$$\begin{aligned} |\psi B_1\rangle &= \hat{\sigma}_2 |\psi\rangle^* \\ |\psi B_2\rangle &= i\hat{\sigma}_2 |\psi\rangle^* \\ |\psi B_3\rangle &= i|\psi\rangle \end{aligned} \tag{4}$$

(Bivectors $\{B_j/2\}$ form a representation of $su(2)$: $[B_i, B_j] = -2\varepsilon_{ijk} B_k$)

3+1D: Lorentz transformations and real spinors

Minkowski spacetime $E^{1,3} = span\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$, $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}$

$\rightarrow \mathcal{C}\ell(E^{1,3}) = span\{1,\ \gamma_\mu,\ \gamma_\mu \wedge \gamma_\nu,\ \gamma_\mu I,\ I\}$, $I = \gamma_0\gamma_1\gamma_2\gamma_3$ $\quad$ (dim 16)

Lorentz transformation (proper, orthochronous):

$$v \ \mapsto\ U\, v\, \widetilde{U} \quad , \quad (a \dots b)^{\sim} = b \dots a \tag{5}$$

(Typically: $U = e^{-B/2}$, where $B = \frac{1}{2} B^{\mu\nu} \gamma_\mu \wedge \gamma_\nu$)

Spinors: $\Psi \in \mathcal{C}\ell_{even}(E^{1,3}) = span\{1, \gamma_\mu \wedge \gamma_\nu, I\} = span\{e^{-B/2}\}$ (dim 8)

(For $V^{p,q}$ define spinors as $\mathcal{C}\ell_{even}(V^{p,q})$ — smallest linear space containing all rotors.)

Real spinor Ψ has 8 *real* components $\leftrightarrow$ 4 *complex* numbers

$\rightarrow$ **Dirac spinor**

$$|\Psi\rangle \equiv \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{C}^4 \quad , \qquad \begin{aligned} z_0 &= \langle \Psi(1 - i\gamma_2\gamma_1)\rangle \\ z_1 &= \langle \gamma_1\gamma_3\Psi(1 - i\gamma_2\gamma_1)\rangle \\ z_2 &= \langle \gamma_3\gamma_0\Psi(1 - i\gamma_2\gamma_1)\rangle \\ z_3 &= \langle \gamma_1\gamma_0\Psi(1 - i\gamma_2\gamma_1)\rangle \end{aligned} \tag{6}$$

Algebraic operations on Ψ $\leftrightarrow$ Matrix operations on $|\Psi\rangle$:

$$\begin{aligned} |\gamma_\mu \Psi \gamma_0\rangle &= \hat{\gamma}_\mu |\Psi\rangle \\ |\gamma_\mu \gamma_\nu \Psi\rangle &= \hat{\gamma}_\mu \hat{\gamma}_\nu |\Psi\rangle \\ \\ |\Psi \gamma_2\gamma_1\rangle &= i|\Psi\rangle \ , \ \ldots \end{aligned} \tag{7}$$

where $\hat{\gamma}_\mu$ are Dirac γ-matrices in *standard* representation:

$$\hat{\gamma}_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} , \ \hat{\gamma}_1 = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} , \ \hat{\gamma}_2 = \begin{pmatrix} & & & i \\ & & -i & \\ & -i & & \\ i & & & \end{pmatrix} , \ \hat{\gamma}_3 = \begin{pmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}$$

Dirac equation - flat spacetime

Matrix Dirac equation (for Dirac spinors) (Dirac 1928):

$$i\hat{\gamma}^{\mu}\partial_{\mu}|\Psi\rangle - m|\Psi\rangle = 0 \tag{8}$$

Real Dirac equation (for real spinors) (Hestenes 1966):

$$\gamma^{\mu}(\partial_{\mu}\Psi)\gamma_0\gamma_2\gamma_1 - m\Psi = 0 \tag{9}$$

Lorentz invariance: $x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$, where $(\Lambda^{-1})^{\nu}{}_{\mu}\gamma^{\mu} = U\gamma^{\nu}\widetilde{U}$

$\rightarrow$ Transformation

$$\gamma'_{\mu} = \gamma_{\mu} \quad , \quad \Psi'(x') = U\Psi(x) \tag{10}$$

does not respect that real spinors Ψ are polynomias in γ_{μ}.

Dirac equation - curved spacetime

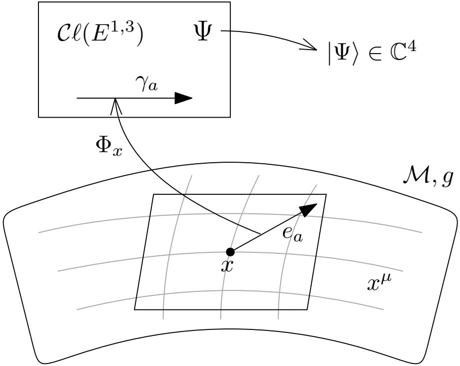
Manifold $\mathcal{M}$ with metric $g_{\mu\nu}$

Tetrad field e_a^μ

Φ_x : tangent space $\to$ flat space

Local Lorentz (gauge) transf.:

$$\gamma_a' = \Phi_x'(e_a) = U\gamma_a\widetilde{U}\ ,\ \ \Psi' = U\Psi\widetilde{U}$$



Real Dirac equation in curved spacetime:

$$\gamma^a e_a^\mu (D_\mu \Psi)\gamma_0\gamma_2\gamma_1 - m\Psi = 0,$$

$$\hookrightarrow D_\mu\Psi = \partial_\mu\Psi - \omega_\mu\Psi + \Psi\mathcal{A}_\mu \qquad (11)$$

Gauge fields ω_μ, $\mathcal{A}_\mu$ are bivector-valued 1-forms that transform as:

$$\omega_\mu' = U\omega_\mu\widetilde{U} + (\partial_\mu U)\widetilde{U}\ ,\ \ \mathcal{A}_\mu' = U\mathcal{A}_\mu\widetilde{U} + (\partial_\mu U)\widetilde{U}$$

Dirac equation - matrix form

The operation $|\ \rangle$ takes complex components:

$$i\hat{\gamma}^a e_a^\mu |D_\mu \Psi\rangle - m|\Psi\rangle = 0 \tag{12}$$

where the covariant derivative translates as

$$|D_\mu \Psi\rangle = \partial_\mu |\Psi\rangle + \frac{\omega_\mu^{ab}}{2} \hat{\gamma}_a \hat{\gamma}_b |\Psi\rangle - \frac{\mathcal{A}_\mu^{ab}}{2} |\Psi \gamma_a \gamma_b\rangle \tag{13}$$

$|\Psi\gamma_2\gamma_1\rangle = i|\Psi\rangle$ $\rightarrow \mathcal{A}_\mu^{12} = qA_\mu \ldots$ elmag. potential

$|\Psi\gamma_3\gamma_0\rangle = \hat{\gamma}_5|\Psi\rangle$

$|\Psi\gamma_1\gamma_0\rangle = -i\hat{\gamma}_2|\Psi\rangle^*$

$|\Psi\gamma_2\gamma_0\rangle = \hat{\gamma}_2|\Psi\rangle^*$

$|\Psi\gamma_3\gamma_2\rangle = -\hat{\gamma}_2\hat{\gamma}_5|\Psi\rangle^*$

$|\Psi\gamma_1\gamma_3\rangle = -i\hat{\gamma}_2\hat{\gamma}_5|\Psi\rangle^*$

(D. Hestenes, arXiv:0807.0060v1
– about electroweak theory)

Here, $\hat{\gamma}_5 = i\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3$,

and * denotes component-wise complex conjugation.

Further remarks

- Gauge field strength:

$$[D_\mu, D_\nu]\Psi = -\mathcal{R}_{\mu\nu}\Psi + \Psi\mathcal{F}_{\mu\nu} \tag{14}$$

where

$$\begin{aligned} \mathcal{R}_{\mu\nu} &= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu - [\omega_\mu, \omega_\nu] \\ \mathcal{F}_{\mu\nu} &= \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - [\mathcal{A}_\mu, \mathcal{A}_\nu] \end{aligned} \tag{15}$$

- Yang-Mills equations (flat spacetime: $e_a^\mu = \delta_a^\mu$, $\omega_\mu = 0$):

$$\partial_\mu\mathcal{F}^{\mu\nu} - [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}] = 0 \tag{16}$$

- For invertible real spinors Ψ ($\Psi = \sqrt{\rho}\, e^{I\beta/2}R$),

$$\gamma^\mu(\partial_\mu\Psi)\gamma_2\gamma_1\Psi^{-1} + q\gamma^\mu A_\mu - m\Psi\gamma_0\Psi^{-1} = 0 \tag{17}$$

allows to express A_μ in terms of Ψ and $\partial_\mu\Psi$.

(See also C. J. Radford, J. Math. Phys. **37**, 4418 (1996), arXiv:hep-th/9510065)

Summary and discussion

- Spinors ψ and Ψ as even elements of real Clifford algebra (relation with Pauli and Dirac spinors $|\psi\rangle$ and $|\Psi\rangle$)
- Interpreting i as a spatial bivector (B_3 or $\gamma_2\gamma_1$)
- In principle, "$i(x)$" can vary throughout spacetime
- Non-Abelian connection $\mathcal{A}_\mu$ acting on the right of Ψ, (generalizing the electromagnetic potential A_μ)

Thank you for your attention.

V. Z., *Real spinors and real Dirac equation*, arXiv:1908.04590 (2019)

3D: Relation with Pauli spinors

Hermitian conjugate: $\langle\Psi| = (z_0^*, z_1^*)$

$\rightarrow Re\langle\Phi|\Psi\rangle = \langle\widetilde{\Phi}\Psi\rangle$ (reversion: $\widetilde{a\dots b} = b\dots a$, e.g. $\widetilde{B}_j = -B_j$)

$$\langle\Phi|\Psi\rangle = Re\langle\Phi|\Psi\rangle - i\,Re(i\langle\Phi|\Psi\rangle) = \langle\widetilde{\Phi}\Psi(1 - i\,B_3)\rangle \qquad (18)$$

Spin "vector":

$$\begin{aligned}\langle\Psi|\hat{\sigma}_j|\Psi\rangle &= \langle\Psi|i\hat{\sigma}_j(-i)|\Psi\rangle = -\langle\Psi|B_j\Psi B_3\rangle = \langle\widetilde{\Psi}\widetilde{B}_j\Psi B_3\rangle \\ &= \langle\widetilde{B}_j\Psi B_3\widetilde{\Psi}\rangle\end{aligned} \qquad (19)$$

$\rightarrow$ Bivector $\Psi B_3\widetilde{\Psi} = \rho R B_3 R^{-1}$ (B_3 rotated by R and stretched by ρ)

$$\Psi B_3\widetilde{\Psi} = \langle\Psi|\hat{\sigma}_j|\Psi\rangle\, B_j \qquad (20)$$

3+1D: Dirac current and other observables

Hermitian conjugate: $\langle\Psi| = (z_0^*, z_1^*, z_2^*, z_3^*) \;\rightarrow\; Re\langle\Phi|\Psi\rangle = \langle\widetilde{\Phi}\gamma_0\Psi\gamma_0\rangle$

Dirac conjugate: $\langle\bar{\Psi}| = \langle\Psi|\hat{\gamma}_0 \;\rightarrow\; Re\langle\bar{\Phi}|\Psi\rangle = Re\langle\Phi|\gamma_0\Psi\gamma_0\rangle = \langle\widetilde{\Phi}\Psi\rangle$

$$\Rightarrow \langle\bar{\Phi}|\Psi\rangle = \langle\widetilde{\Phi}\Psi(1 - i\gamma_2\gamma_1)\rangle \tag{21}$$

Dirac current:

$$\langle\bar{\Psi}|\hat{\gamma}_\mu|\Psi\rangle = \langle\bar{\Psi}|\gamma_\mu\Psi\gamma_0\rangle = \langle\widetilde{\Psi}\gamma_\mu\Psi\gamma_0\rangle = \langle\gamma_\mu\Psi\gamma_0\widetilde{\Psi}\rangle \tag{22}$$

(Canonical form $\Psi = \sqrt{\rho}e^{I\beta/2}R$ & $I\gamma_\mu = -\gamma_\mu I$
$\Rightarrow \Psi\gamma_0\widetilde{\Psi} = \rho R\gamma_0\widetilde{R}$ is a vector rotated by R and stretched by ρ)

$$\Psi\gamma_0\widetilde{\Psi} = \langle\bar{\Psi}|\hat{\gamma}_\mu|\Psi\rangle\,\gamma^\mu \tag{23}$$

Other observables:

$$\langle\bar{\Psi}|\tfrac{i}{2}[\hat{\gamma}^\mu, \hat{\gamma}^\nu]|\Psi\rangle = (\gamma^\mu \wedge \gamma^\nu)\cdot(\Psi\gamma_2\gamma_1\widetilde{\Psi}) \quad , \quad \langle\bar{\Psi}|\Psi\rangle = \langle\widetilde{\Psi}\Psi\rangle$$

$$\langle\bar{\Psi}|\hat{\gamma}^\mu\hat{\gamma}_5|\Psi\rangle = \gamma^\mu\cdot(\Psi\gamma_3\widetilde{\Psi}) \quad , \quad \langle\bar{\Psi}|i\hat{\gamma}_5|\Psi\rangle = \langle\Psi\widetilde{\Psi}I\rangle$$