Local time path integrals and their application to Lévy random walks

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Leiden

Motivation

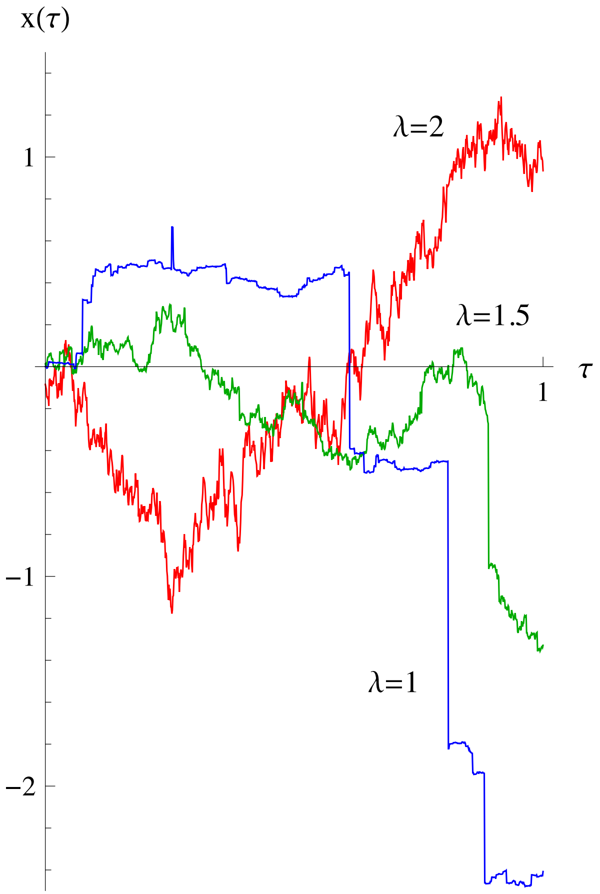
TIME ZONES:

Motivation

TIME ZONES: NOT INCLUDED IN THIS PRESENTATION

Motivation

Local time of a stochastic process:



Motivation

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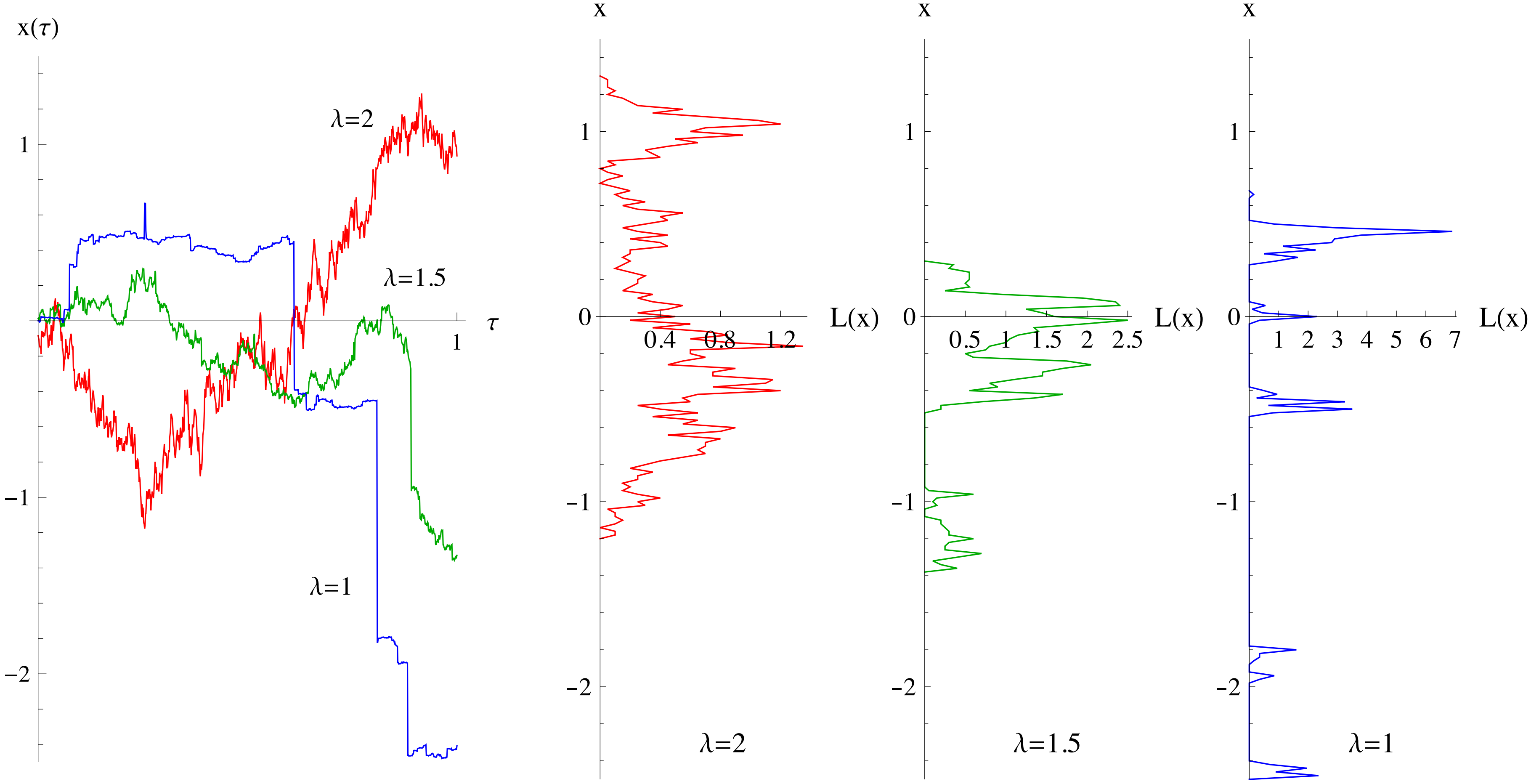


Stochastic trajectory $x(\tau)$ $\rightarrow$ Local time profile $L(x)$

Outline

- Preliminaries: diffusion equation and path integral
- Introduction of local time of a stochastic process
- Correlation functions and functionals of the local time
- Time-independent systems and the resolvent method
- Local time of Lévy random walks (Lévy flights)
- Local-time representation for Gaussian path integrals

In collaboration with Petr Jizba (Czech Technical University in Prague) and Angel Alastuey (ENS Lyon).

Diffusion equation and path integral

Diffusion (or heat, or Fokker-Planck) equation

$$[\partial_t + H(-i\partial_x, x, t)]\, P(x,t) = 0 \tag{1}$$

where $H(p,x,t)$ is a generic Hamiltonian, and $x \in \mathbb{R}$. $[\hbar = 1]$

Solution for initial condition $P(x,t_a) = \delta(x - x_a)$ represented by phase-space path integral

$$(x_b t_b | x_a t_a) \equiv P(x_b, t_b) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \int \frac{\mathcal{D}p}{2\pi}\, e^{\mathcal{A}[p,x]} \tag{2}$$

where the action

$$\mathcal{A}[p,x] = \int_{t_a}^{t_b} d\tau \left[ip(\tau)\dot{x}(\tau) - H(p(\tau), x(\tau), \tau) \right] \tag{3}$$

Diffusion equation and path integral

We will gradually specify the Hamiltonian:

$H(p, x, t)$ — generic, possibly time-dependent

$\cup$

$H(p, x)$ — time-independent

$\cup$

$H(p)$ — time and position-independent

$\cup$

$H_\lambda(p) = D_\lambda (p^2)^{\lambda/2}$ — Lévy Hamiltonian

In the end, special attention to

$H(p, x) = \frac{p^2}{2M} + V(x)$ — usual quantum Hamiltonian

Diffusion equation and path integral

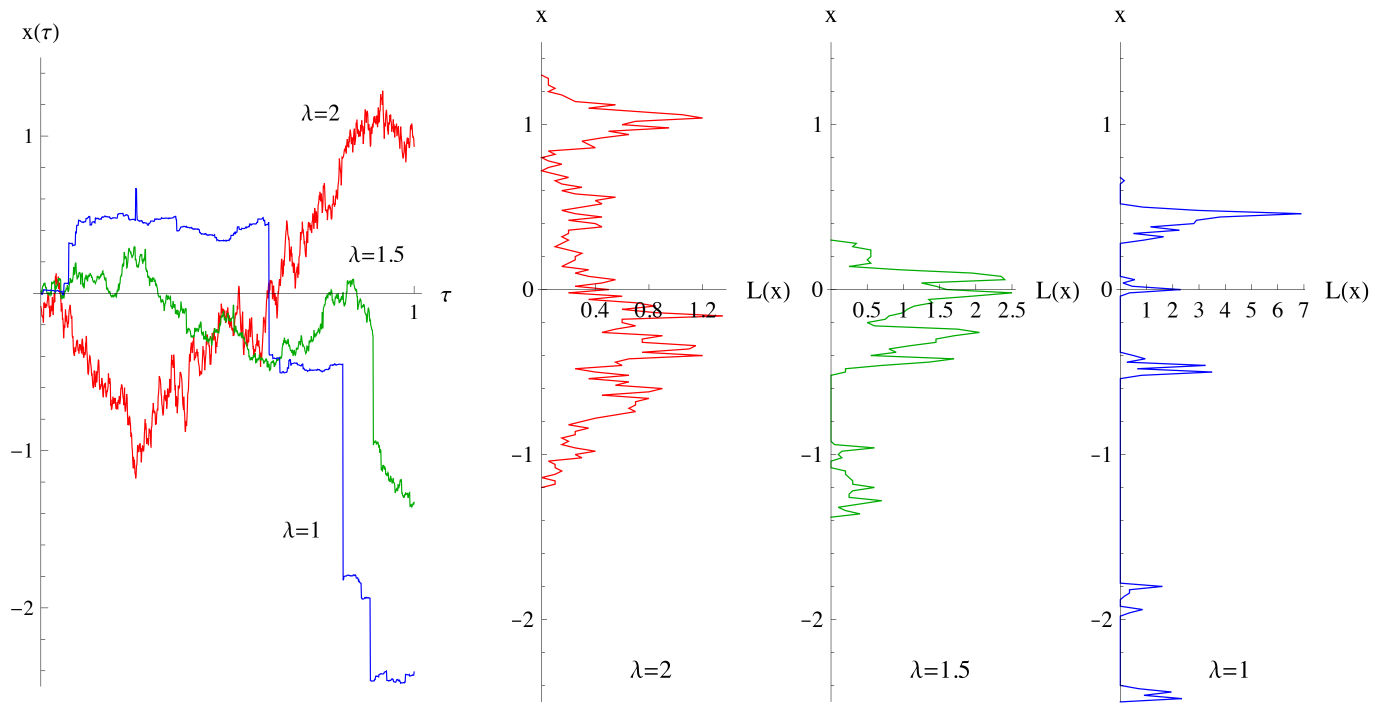
Significance of the heat equation:

- Diffusion processes (or continuous random walks)
  – $P(x,t)$ is the probability of transition to point x after time t
- Quantum mechanics: by rotation to imaginary times $t \to it$
  – P are transition amplitudes
- Quantum statistical physics: identifying $t = \beta\hbar$, where $\beta = \frac{1}{k_B T}$
  – P are matrix elements of the equilibrium density matrix
- ...

In addition, corresponding path integrals describe:

- Polymers or other line-like objects
- Stock prices on financial markets
- ...

Local time of a stochastic process



For stochastic trajectory $x(\tau)$ define

$$L(x; t_a, t_b, x(\tau)) = \int_{t_a}^{t_b} d\tau\, \delta(x - x(\tau)) \qquad (4)$$

P. Lévy, *Sur certains processus stochastiques homogènes*, Compos. Math. **7**, 283 (1939).

Local time of a stochastic process

L is functional of $x(\tau)$ and function of x:

- $L(x) \geq 0$
- $\int_{\mathbb{R}} L(x)dx = t_b - t_a$
- $L(x)$ has compact support
- $L(x)$ is continuous

$L(x) \ldots$ trajectories of a new stochastic process

Correlation functions of local time

$$\langle L(x_1)\dots L(x_n)\rangle = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \int \frac{\mathcal{D}p}{2\pi}\, e^{\mathcal{A}[p,x]} L(x_1)\dots L(x_n)$$

$$= \sum_{\sigma\in S_n} \int_{t_a<t_1<\dots<t_n<t_b} dt_1\dots dt_n \prod_{k=0}^{n} (x_{\sigma(k+1)}t_{k+1}|x_{\sigma(k)}t_k) \qquad (5)$$

where σ are permutations of indices $\{1,\dots,n\}$, with $\sigma(0)=0$ and $\sigma(n+1)=n+1$, and we have denoted $x_0=x_a$, $t_0=t_a$, $x_{n+1}=x_b$, $t_{n+1}=t_b$.

Generic functionals of local time

$L(x_1)\dots L(x_n)$ is an example of functional of the local time $F[L(x)]$.
In general,

$$\langle F[L(x)]\rangle = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \int \frac{\mathcal{D}p}{2\pi}\, e^{\mathcal{A}[p,x]} F[L(x)]$$

$$= F\left[-\frac{\delta}{\delta U(x)}\right] \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \int \frac{\mathcal{D}p}{2\pi}\, e^{\mathcal{A}_U[p,x]}\Big|_{U=0} \tag{6}$$

where $\mathcal{A}_U[p,x]$ corresponds to $H_U(p,x,t) = H(p,x,t) + U(x)$.

We have employed the key identity

$$\int_{t_a}^{t_b} U(x(\tau))\, d\tau = \int_{-\infty}^{\infty} U(x)L(x)\, dx \tag{7}$$

Time-independent systems and the resolvent method

Assume $H(p, x)$ [independent of time t]
$\Rightarrow \ (x_b t_b | x_a t_a) = \langle x_b | e^{-(t_b - t_a)\hat{H}} | x_a \rangle \ \rightarrow$ take $t_a = 0$, $t_b \equiv t$ from now on.

Laplace representation $\langle F[L(x)] \rangle_E = \int_0^\infty dt \, e^{-tE} \langle F[L(x)] \rangle$

Correlation functions:

$$\langle L(x_1) \ldots L(x_n) \rangle_E = \sum_{\sigma \in S_n} \prod_{k=0}^{n} R(x_{\sigma(k)}, x_{\sigma(k+1)}, -E) \tag{8}$$

[e.g. $\langle L(x) \rangle_E = R(x_a, x) R(x, x_b)$]

Here

$$R(x, x', -E) \equiv \langle x' | (\hat{H} + E)^{-1} | x \rangle \tag{9}$$

denotes matrix elements of the resolvent operator $\hat{R}(E) = (\hat{H} - E)^{-1}$.

Time-independent systems and the resolvent method

Generic functionals:

$$\langle F[L(x)]\rangle_E = F\left[-\frac{\delta}{\delta U(x)}\right] R_U(x_a, x_b, -E)\bigg|_{U=0}, \tag{10}$$

where $\hat{R}_U(E) = (\hat{H}_U - E)^{-1}$ is the resolvent corresponding to the extended Hamiltonian $H_U = H + U$.

One-point distribution: take $U(x') = u\,\delta(x' - x) \;\to\; \frac{\delta}{\delta U(x)} = \frac{\partial}{\partial u}$

$$\ldots \;\Rightarrow\; R_U(x_a, x_b) = R(x_a, x_b) - u\frac{R(x_a, x)R(x, x_b)}{1 + uR(x, x)} \tag{11}$$

$$\begin{aligned}\big\langle \delta(L - L(x))\big\rangle_E = \theta(L)\frac{R(x_a, x)R(x, x_b)}{R(x, x)^2} e^{-\frac{L}{R(x,x)}} \\ + \delta(L)\left[R(x_a, x_b) - \frac{R(x_a, x)R(x, x_b)}{R(x, x)}\right]\end{aligned} \tag{12}$$

* Multipoint resolvents

Resolvent identity

$$(\hat{H}_U + E)^{-1} = (\hat{H} + E)^{-1} - (\hat{H}_U + E)^{-1} U(\hat{x})(\hat{H} + E)^{-1} \tag{13}$$

for

$$U(x) = \sum_{j=1}^{n} u_j \delta(x - x_j) \tag{14}$$

reads

$$R_U(x_a, x_b) = R(x_a, x_b) - \sum_{j=1}^{n} u_j R(x_a, x_j) R_U(x_j, x_b) \tag{15}$$

Setting $x_a = x_k$ for $k = 1, \dots, n$,

$$R_U(x_j, x_b) = \sum_{k=1}^{n} M_{jk}^{-1} R(x_k, x_b) \tag{16}$$

where M_{jk}^{-1} are elements of the inverse of the matrix

$$M_{jk} = \delta_{jk} + u_k R(x_j, x_k) \tag{17}$$

Distribution functions and moments of local time

Invert the Laplace transform to return to the time domain: $\langle \dots \rangle_E \to \langle \dots \rangle$

Moments of local time:

$$\mu(x_1, \dots, x_n) = \frac{\langle L(x_1) \dots L(x_n) \rangle}{\langle 1 \rangle} \tag{18}$$

n-point distribution functions of local time:

$$W(L_1, \dots, L_n; x_1, \dots, x_n) = \frac{\left\langle \prod_{j=1}^{n} \delta(L_j - L(x_j)) \right\rangle}{\langle 1 \rangle} \tag{19}$$

The normalization factor $\langle 1 \rangle$ equals $(x_b t_b | x_a t_a)$ for averaging over paths with fixed initial point x_a and final point x_b.

Averaging over paths with arbitrary x_b: $\langle \dots \rangle^* \equiv \int_{-\infty}^{\infty} dx_b \, \langle \dots \rangle$
$\to \langle 1 \rangle^*, \mu^*, W^*, \dots$

Lévy random walks

Hamiltonians $H(p)$ only dependent on momentum form now on:

$$P(x,t) \equiv (x_b t|x_a 0) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-tH(p)} e^{ipx} \quad , \quad x = x_b - x_a \qquad (20)$$

For the Lévy Hamiltonian $H_\lambda(p) = D_\lambda (p^2)^{\lambda/2}$ [we consider $\lambda \in [1,2]$]
$\rightarrow$ symmetric Lévy stable distribution

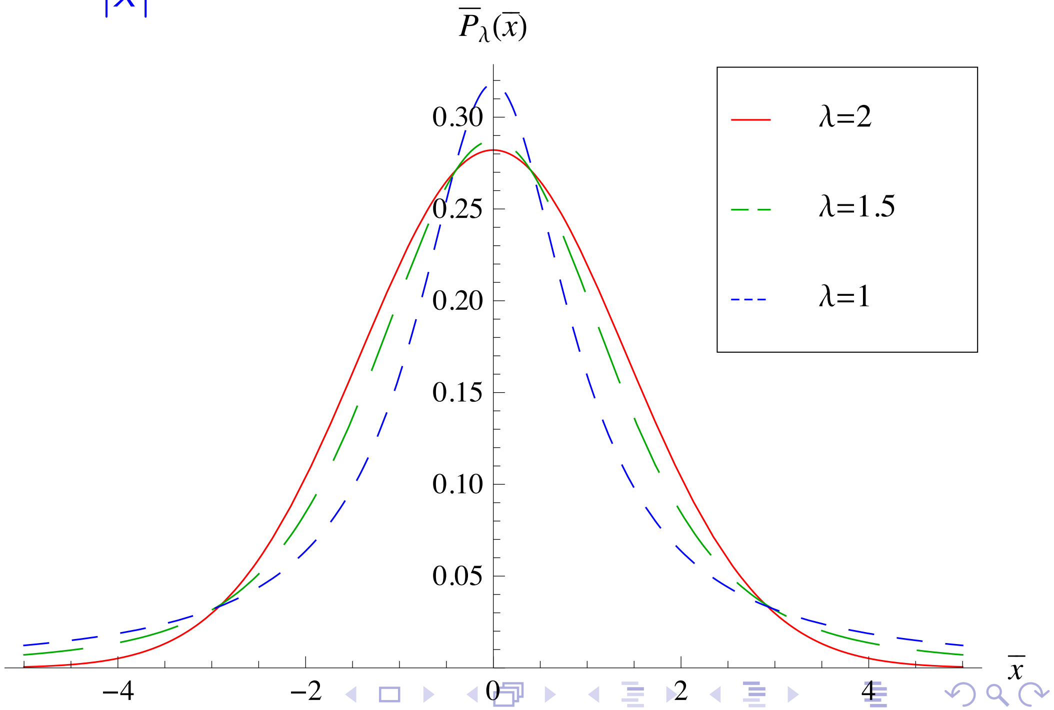
Heavy tails for $\lambda < 2$: $P_\lambda(x,t) \overset{|x|\to\infty}{\sim} \frac{c_\lambda}{|x|^{1+\lambda}}$

For $\lambda = 2$ Gaussian:

$$P_{\lambda=2}(x,t) = \frac{e^{-\frac{x^2}{4D_2 t}}}{\sqrt{4\pi D_2 t}}$$

For $\lambda = 1$ Cauchy(-Lorentz):

$$P_{\lambda=1}(x,t) = \frac{1}{\pi} \frac{D_1 t}{(D_1 t)^2 + x^2}$$



Lévy stable distributions

Stability: $X_1 \sim Levy \quad , \quad X_2 \sim Levy \quad \Rightarrow \quad X_1 + X_2 \sim Levy$

Characteristic function:

$$\exp\left[ip\mu - |cp|^\lambda\left(1 - i\beta sgn(p)\Phi\right)\right] \quad , \quad \Phi = \begin{cases} \tan(\lambda\pi/2) & \lambda \neq 1 \\ -(2/\pi)\log|p| & \lambda = 1 \end{cases}$$

$\lambda \ldots$tail power ($\sim |x|^{-1-\lambda}$), $c \ldots$width, $\mu \ldots$shift of origin, $\beta \ldots$asymmetry

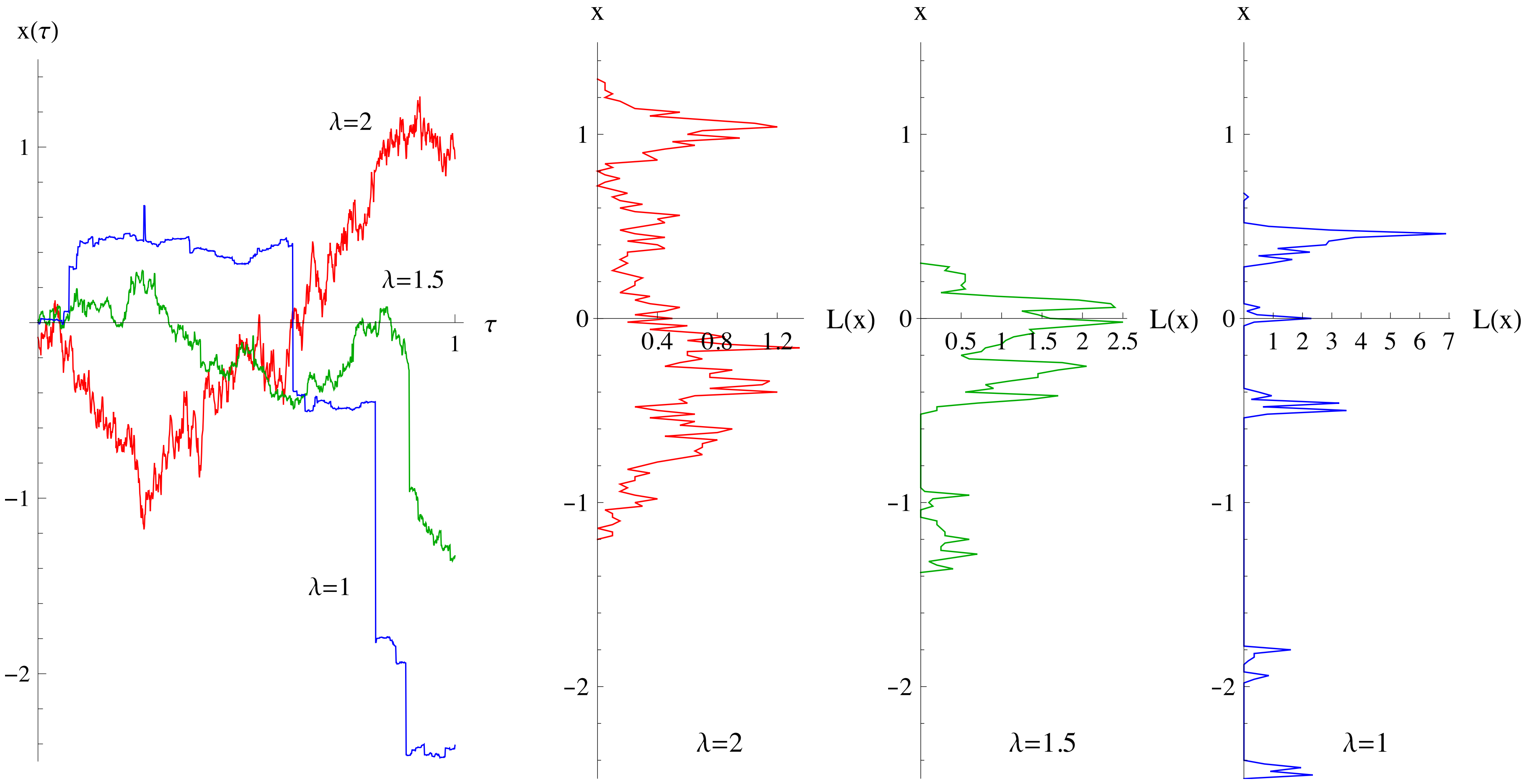
Generalized central limit theorem:

$$i.i.d. \quad X_1, \ldots, X_n \quad \sim \quad \text{“}\textit{distribution with possibly infinite variance}\text{”}$$

$$\lim_{n\to\infty} \frac{X_1 + \ldots + X_n}{\mathcal{N}_n} \quad \sim \quad Levy$$

Applications in financial markets (fluctuation of stock prices), physics (e.g. François Bardou et al., *Lévy Statistics and Laser Cooling: How Rare Events Bring Atoms to Rest*, CUP (2002)., . . .), . . .

Lévy random walks



For decreasing λ:

Longer jumps (Lévy flights)

Separation of local time $L(x)$ into peaks

Resolvent of Lévy Hamiltonian

$$R_\lambda(x,x',-E) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{e^{ip(x'-x)}}{D_\lambda (p^2)^{\lambda/2} + E} \qquad (21)$$

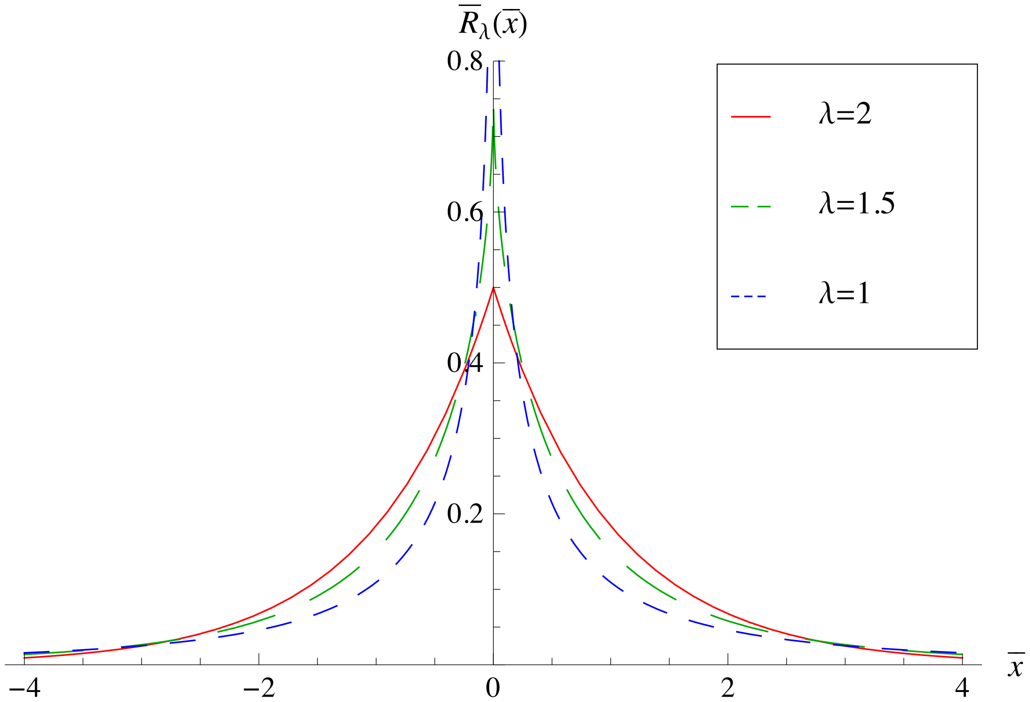
... Linnik (or geometric stable) distribution

For $\lambda = 2$ "Gaussian" resolvent:

$$R_2(x,-E) = \frac{e^{-\sqrt{E/D_2}|x|}}{\sqrt{4D_2E}}$$

For $\lambda = 1$ "Cauchy" resolvent:

$$R_1(x,-E) = \frac{-1}{\pi D_1}\left[\sin\frac{E|x|}{D_1}\,\mathrm{si}\frac{E|x|}{D_1} + \cos\frac{E|x|}{D_1}\,\mathrm{ci}\frac{E|x|}{D_1}\right]$$



divergent for $\lambda = 1$ at $x = 0$

One-point distributions of local time at the origin

1) $x_b = x_a$: $\left\langle \delta(L - L(x_a)) \right\rangle_E = e^{-\frac{L}{R_\lambda(0)}} = \exp\left[-\sigma_\lambda L E^{1-\frac{1}{\lambda}}\right]$, $\sigma_\lambda \equiv \lambda\, D_\lambda^{1/\lambda} \sin\frac{\pi}{\lambda}$

$$W_\lambda(L; x_a) = \frac{\langle \delta(L - L(x_a)) \rangle}{\langle 1 \rangle} \tag{22}$$

$$= \frac{\lambda (D_\lambda t)^{1/\lambda}}{\Gamma(\frac{1}{\lambda})} \int_0^\infty dE\, e^{-Et} \exp\left[L E^{1-\frac{1}{\lambda}} \sigma_\lambda \cos\frac{\pi}{\lambda}\right] \sin\left[L E^{1-\frac{1}{\lambda}} \sigma_\lambda \sin\frac{\pi}{\lambda}\right]$$

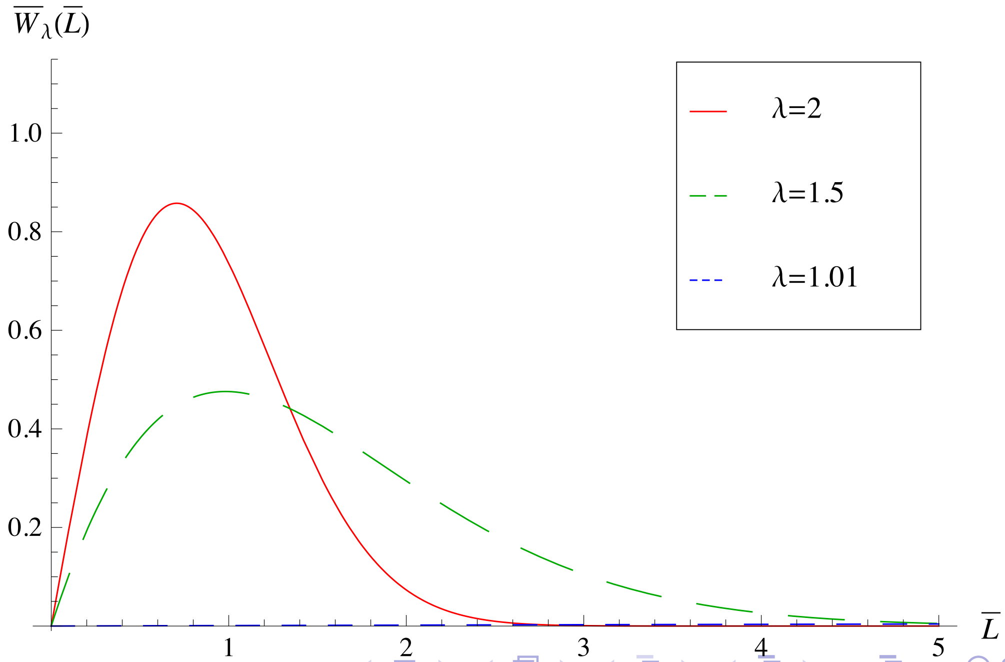
Gaussian case:

$$W_{\lambda=2}(L; x_a) = \frac{2 D_2 L}{t} e^{-D_2 L^2/t}$$

For $\lambda = 1$ (Cauchy case):

singularity $R_\lambda(0) = +\infty$

$\rightarrow$ flattening of the distribution



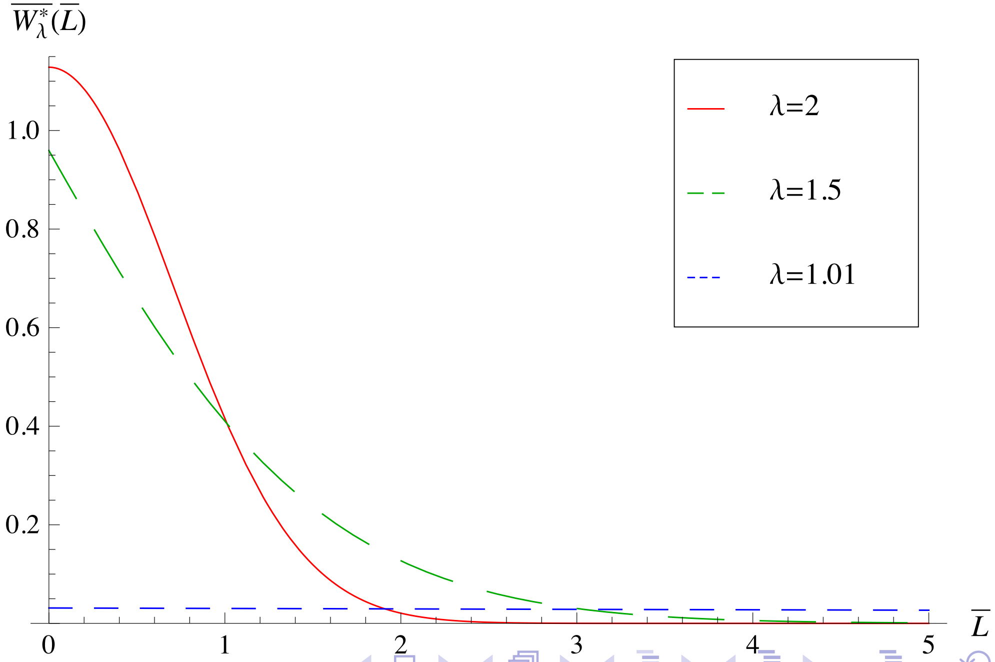
2) free x_b: $\left\langle \delta(L - L(x_a)) \right\rangle_E^* = \frac{1}{ER_\lambda(0)} e^{-\frac{L}{R_\lambda(0)}} = \frac{\sigma_\lambda}{E^{1/\lambda}} \exp\left[-\sigma_\lambda L E^{1-\frac{1}{\lambda}}\right]$

$$W_\lambda^*(L;x_a) = \frac{\left\langle \delta(L - L(x_a)) \right\rangle^*}{\langle 1 \rangle^*} \tag{23}$$

$$= \frac{\sigma_\lambda}{\pi} \int_0^\infty dE \, \frac{e^{-Et}}{E^{1/\lambda}} \exp\left[L E^{1-\frac{1}{\lambda}} \sigma_\lambda \cos\frac{\pi}{\lambda}\right] \sin\left[L E^{1-\frac{1}{\lambda}} \sigma_\lambda \sin\frac{\pi}{\lambda} + \frac{\pi}{\lambda}\right]$$

Gaussian case:

$$W_{\lambda=2}^*(L;x_a) = \sqrt{\frac{4D_2}{\pi t}} \, e^{-D_2 L^2/t}$$



* One-point distribution of Brownian local time

Gaussian case $\lambda = 2$ (Brownian motion): generic x_b and x

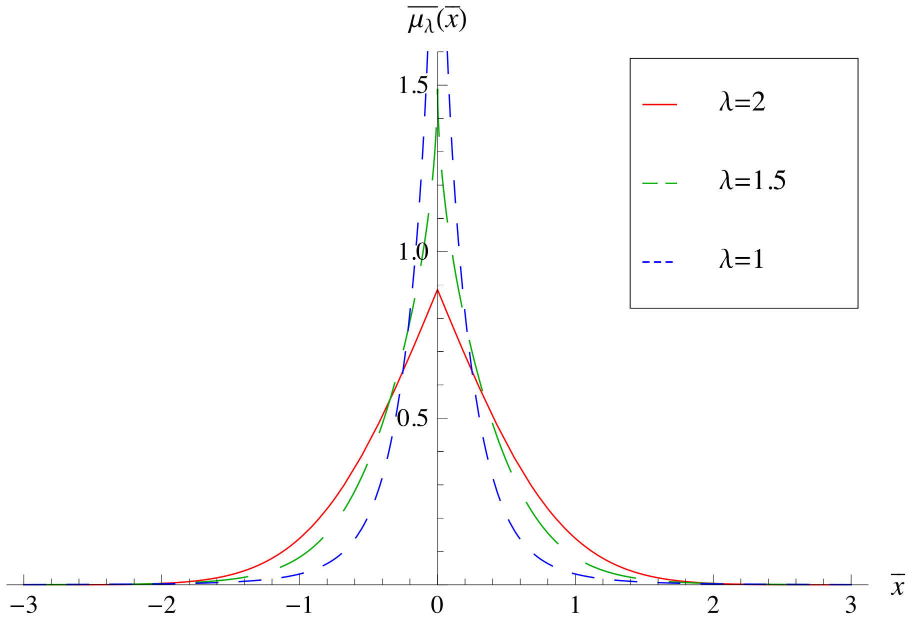
$$W_2(L;x) = \theta(L)\frac{|x_a - x| + |x - x_b| + 2D_2 L}{t} \times \exp\left[-\frac{(|x_a - x| + |x - x_b| + 2D_2 L)^2 - (x_b - x_a)^2}{4D_2 t}\right] + \delta(L)\left\{1 - \exp\left[-\frac{(|x_a - x| + |x - x_b|)^2 - (x_b - x_a)^2}{4D_2 t}\right]\right\} \quad (24)$$

$$W_2^*(L;x) = \theta(L)\sqrt{\frac{4D_2}{\pi t}}\exp\left[-\frac{(|x_a - x| + 2D_2 L)^2}{4D_2 t}\right] + \delta(L)\,\mathrm{erf}\left[\frac{|x_a - x|}{\sqrt{4D_2 t}}\right] \quad (25)$$

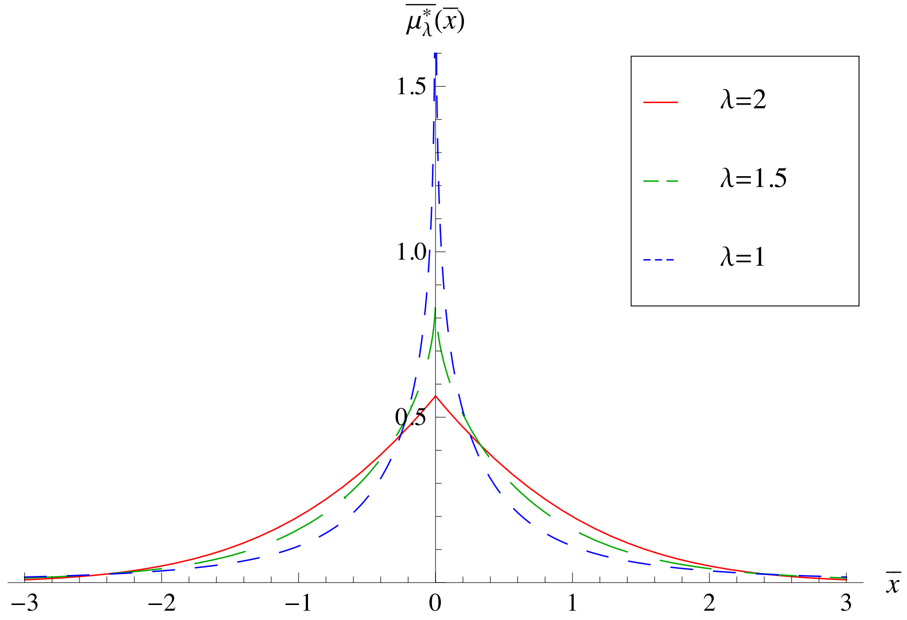
The δ-term vanishes for x between x_a and x_b.

First moments of local time

[$x_b = x_a$]



[free x_b]



$$\mu_{\lambda=2}(x) = \sqrt{\frac{\pi t}{4D_2}}\, e^{\frac{(x_b - x_a)^2}{4D_2 t}}\, \mathrm{erfc}\left[\frac{|x_a - x| + |x - x_b|}{\sqrt{4D_2 t}}\right]$$

$$\mu^*_{\lambda=2}(x) = \sqrt{\frac{t}{\pi D_2}}\, e^{-\frac{(x_a - x)^2}{4D_2 t}} - \frac{|x_a - x|}{2D_2}\, \mathrm{erfc}\left[\frac{|x_a - x|}{\sqrt{4D_2 t}}\right]$$

$$\mu_{\lambda=1}(x) = \frac{-1}{2\pi D_1^2 t\left[(D_1 t)^2 + (x_a + x_b - 2x)^2\right]}\Big\{2(x - x_a)\left[(D_1 t)^2 + (x - x_a)^2 - (x_b - x)^2\right] \arctan\frac{D_1 t}{x - x_a}$$
$$+2(x_b - x)\left[(D_1 t)^2 + (x_b - x)^2 - (x - x_a)^2\right] \arctan\frac{D_1 t}{x_b - x}$$
$$+D_1 t\left[(D_1 t)^2 + (x - x_a)^2 + (x_b - x)^2\right] \ln\frac{(x - x_a)^2 (x_b - x)^2}{\left[(D_1 t)^2 + (x - x_a)^2\right]\left[(D_1 t)^2 + (x_b - x)^2\right]}\Big\}$$

$$\mu^*_{\lambda=1}(x) = \frac{1}{2\pi D_1} \ln\left[1 + \left(\frac{D_1 t}{x - x_a}\right)^2\right]$$

* Second moments of Brownian local time

1) fixed x_b

$$\mu_2(x_1, x_2) = \frac{t}{2D_2} \left\{ \exp\left[-\frac{\xi_{12}^2 - (x_b - x_a)^2}{4D_2 t} \right] - \frac{\sqrt{\pi}\,\xi_{12}}{\sqrt{4D_2 t}} \mathrm{erfc}\left[\frac{\xi_{12}}{\sqrt{4D_2 t}} \right] \exp\left[\frac{(x_b - x_a)^2}{4D_2 t} \right] \right\} + (\xi_{12} \leftrightarrow \xi_{21})$$

where $\xi_{12} \equiv |x_a - x_1| + |x_1 - x_2| + |x_2 - x_b| \quad , \quad \xi_{21} \equiv |x_a - x_2| + |x_2 - x_1| + |x_1 - x_b|$

2) free x_b

$$\mu_2^*(x_1, x_2) = \frac{1}{4D_2} \left\{ -\frac{\sqrt{t}\,\xi_{12}^*}{\sqrt{\pi D_2}} \exp\left[-\frac{(\xi_{12}^*)^2}{4D_2 t} \right] + \left[\frac{(\xi_{12}^*)^2}{2D_2} + t \right] \mathrm{erfc}\left[\frac{\xi_{12}^*}{\sqrt{4D_2 t}} \right] \right\} + (\xi_{12}^* \leftrightarrow \xi_{21}^*)$$

where $\xi_{12}^* \equiv |x_a - x_1| + |x_1 - x_2| \quad , \quad \xi_{21}^* \equiv |x_a - x_2| + |x_2 - x_1|$

Path integrals in quantum statistical physics

From now on

$$H(p,x) = \frac{p^2}{2M} + V(x) \quad (26)$$

Motivation in quantum statistical physics:

- Gibbs operator: $e^{-\beta \hat{H}}$
  $\rightarrow$ partition function: $Tr(e^{-\beta \hat{H}}) \quad , \quad \beta = 1/k_B T$
  $\rightarrow$ thermal density matrix: $e^{-\beta \hat{H}} / Tr(e^{-\beta \hat{H}})$

$$\rho(x_a, x_b, \beta) \equiv \langle x_b | e^{-\beta \hat{H}} | x_a \rangle$$

- $\rho \ldots$ Heat kernel for diffusion generated by H, with time variable β:

$$\left[\partial_\beta + H(-i\hbar\partial_x, x)\right] \rho(x_a, x, \beta) = 0 \quad (27)$$

Path integrals in quantum statistical physics

1) Feynman path-integral representation

$$\rho(x_a, x_b, \beta) = \int_{x(0)=x_a}^{x(\beta\hbar)=x_b} \mathcal{D}x(\tau) \exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \left[\frac{M}{2}\dot{x}^2(\tau) + V(x(\tau))\right]\right\} \quad (28)$$

For $\beta \to 0 \ldots$ high temperatures:

$$V(x(\tau)) = V(x_a) + V'(x_a)(x(\tau) - x_a) + \ldots \Rightarrow \rho \sim \frac{e^{-\beta V(x_a)}}{\sqrt{2\pi\beta\hbar^2/M}}$$

P. Jizba and V. Zatloukal, *Path-integral approach to the Wigner-Kirkwood expansion*, Phys. Rev. E **89**, 012135 (2014)

2) Spectral representation

$$\rho(x_a, x_b, \beta) = \sum_n e^{-\beta E_n}\psi_n^*(x_a)\psi_n(x_b) \quad \text{where} \quad \hat{H}|\psi_n\rangle = E_n|\psi_n\rangle \quad (29)$$

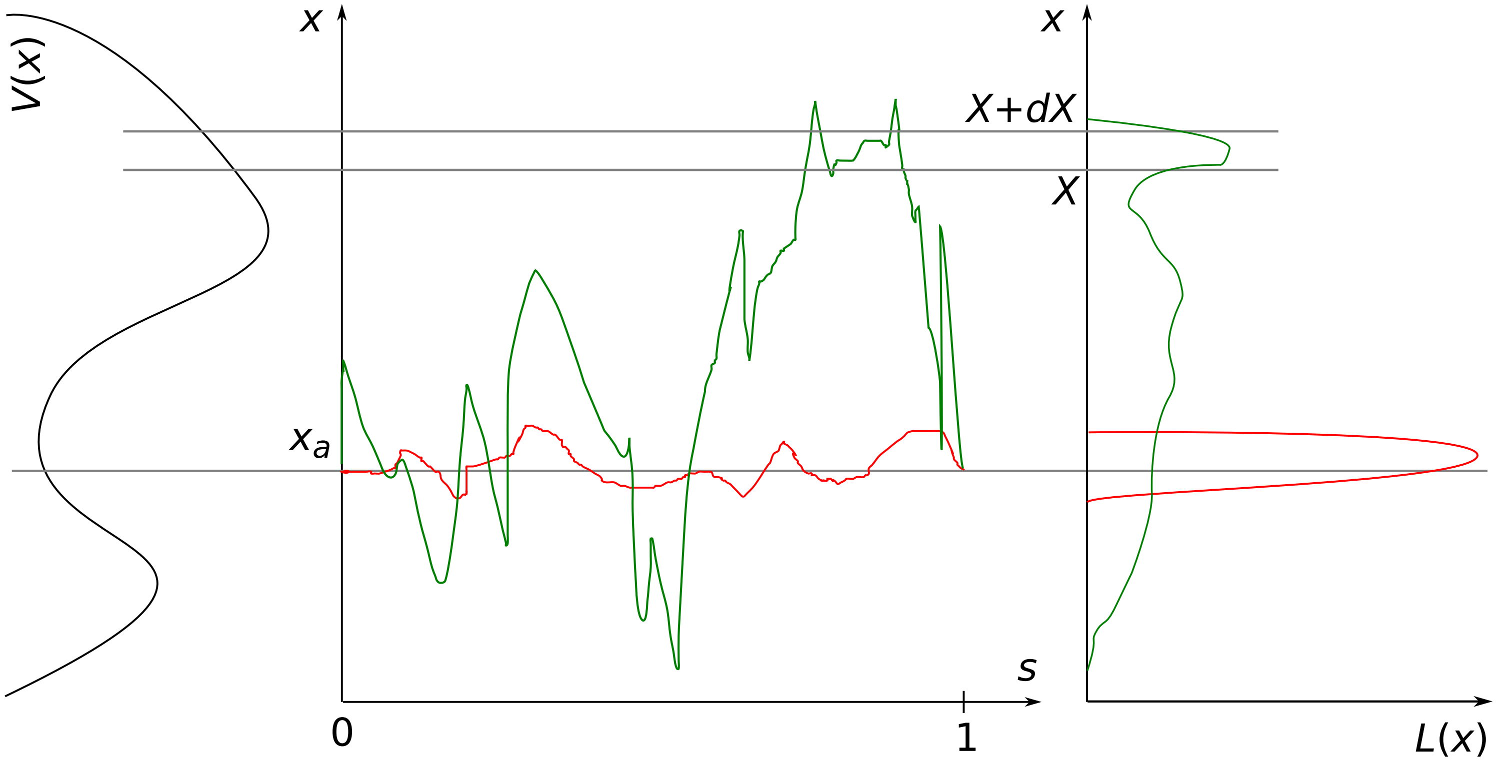
For $\beta \to \infty \ldots$ low temperatures: $\rho \sim e^{-\beta E_{gs}}\psi_{gs}^*(x_a)\psi_{gs}(x_b)$

3) Local-time representation? – path integral over local time profiles

$$L(x) = \int_0^{\beta\hbar} d\tau\, \delta(x - x(\tau)) \tag{30}$$

$\tau \to s = \tau/\beta\hbar$

- small β
- large β



Local-time path integral

$$\rho(x_a, x_b, \beta) = \int_{x(0)=x_a}^{x(\beta\hbar)=x_b} \mathcal{D}x(\tau) \exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar} \frac{M}{2}\dot{x}^2(\tau)d\tau - \frac{1}{\hbar}\int_{\mathbb{R}} V(X)L(X)dX\right\}$$

$$\Downarrow$$

change of variables: $x(\tau) \;\rightarrow\; L(x)$

$$\Downarrow$$

$$\rho(x_a, x_b, \beta) = \int \mathcal{D}L(x) W[L(x); \beta\hbar, x_a, x_b] \exp\left\{-\int_{\mathbb{R}} L(x)V(x)dx\right\}$$

where $L(x) \geq 0$ and $W[L(x)] = 0$ if $\int_{\mathbb{R}} L(x)dx \neq \beta\hbar$

$\rightarrow$ identify the weights $W[L(x)]$

Local-time path integral: glimpses of the derivation

- Diffusion equation in Laplace picture:
  $\left[E + H(-i\hbar\partial_x, x)\right]\widetilde{\rho}(x_a, x, E) = \delta(x_a - x)$

- Quantum-field-theoretic representation:
  $\widetilde{\rho}(x_a, x_b, E) = 2 \int \mathcal{D}\psi(x)\psi(x_a)\psi(x_b)e^{-\langle\psi|E+\hat{H}|\psi\rangle} / \int \mathcal{D}\psi(x)e^{-\langle\psi|E+\hat{H}|\psi\rangle}$

- Replica trick: $a/b = \lim_{D\to 0} ab^{D-1}$

  $\widetilde{\rho} = \lim_{D\to 0} \frac{2}{D} \int \mathcal{D}^D\psi(x)\psi_1(x_a)\psi_1(x_b) \exp\left\{-\sum_{\sigma=1}^{D}\langle\psi_\sigma|E + \hat{H}|\psi_\sigma\rangle\right\}$

- Spherical coordinates in ψ-space: radial part $\eta = \sqrt{\vec{\psi}\cdot\vec{\psi}}$

- Inverse Laplace transform $\Rightarrow$ *Local-time representation of* $\rho(x_a, x_b, \beta)$

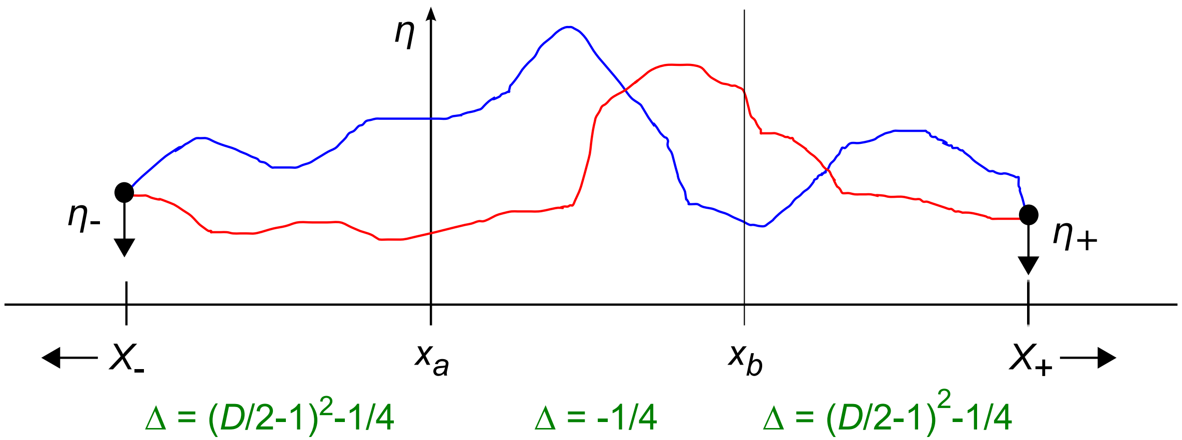
Local-time path integral

$$\rho(x_a,x_b,\beta)\equiv\langle x_b|e^{-\beta\hat{H}}|x_a\rangle=\lim_{X_\pm\to\pm\infty}\lim_{D\to 0}\frac{2}{D^2}\lim_{\eta_\pm\to 0}(\eta_-\eta_+)^{\frac{1-D}{2}}$$
$$\times\int_{\eta(X_-)=\eta_-}^{\eta(X_+)=\eta_+}\mathcal{D}\eta(x)\,\delta\left(\int_{X_-}^{X_+}\eta^2(x)dx-\beta\right)\eta(x_a)\eta(x_b)\exp\left\{-A_\Delta[\eta(x)]\right\}$$

where

$\eta(x)\geq 0$

$\eta^2(x)\leftrightarrow L(x)/\hbar$



The action of a radial harmonic oscillator ($\leftrightarrow$ Bessel process indexed by x)

$$A_\Delta[\eta(x)]\equiv\int_{X_-}^{X_+}dx\left[\frac{\hbar^2}{2M}\eta'(x)^2+V(x)\eta^2(x)+\frac{M}{\hbar^2}\frac{\Delta(x)}{2\eta^2(x)}\right]\quad(31)$$

Local-time path integral at low temperatures

Rescaling $\eta \to \sqrt{\beta}\eta$:

$$A_\Delta[\sqrt{\beta}\eta(x)] = \beta \int_{X_-}^{X_+} dx \left[\frac{\hbar^2}{2M} \eta'(x)^2 + V(x)\eta^2(x) + \frac{M}{\hbar^2} \frac{\Delta(x)}{2\beta^2\eta^2(x)} \right]$$

$\beta \to \infty$:

Saddle-point approximation of the path integral ($\to$ neglect the last term)

$\leftrightarrow$ Minimization of the functional:

$$\int_{X_-}^{X_+} dx \left[\frac{\hbar^2}{2M} \eta'(x)^2 + V(x)\eta^2(x) \right] = \langle \eta | \hat{H} | \eta \rangle$$

under the constraint $\langle \eta | \eta \rangle = 1$

$\Rightarrow$ Rayleigh-Ritz variational principle for the ground state

Generic functionals of the local time

Average value of generic functional $F[L(x)]$:

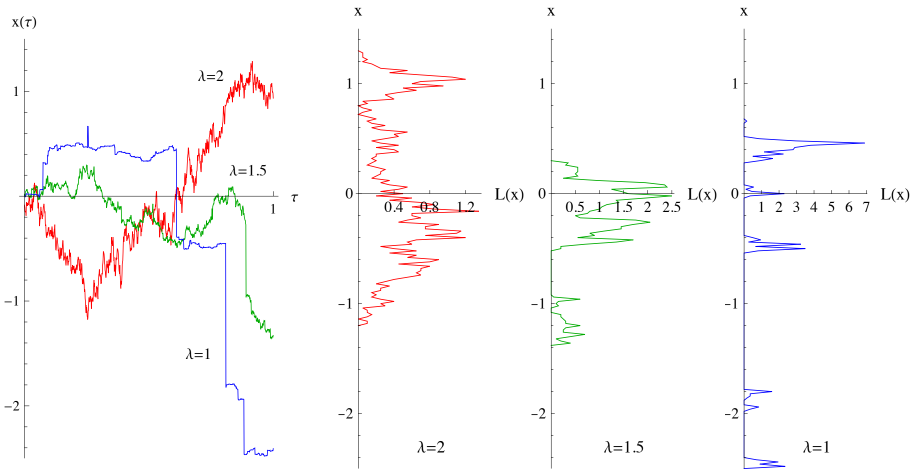
$$\int_{x(0)=x_a}^{x(\beta\hbar)=x_b} \mathcal{D}x(\tau) F[L(X)] \exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \left[\frac{M}{2}\dot{x}^2(\tau) + V(x(\tau))\right]\right\}$$

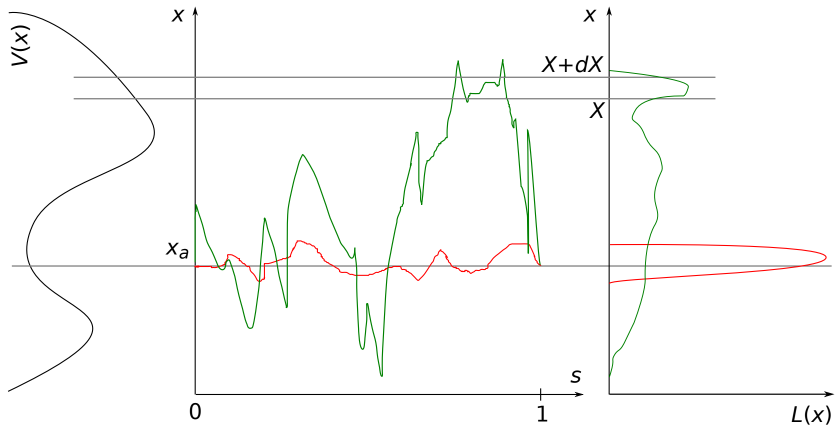
$$= \lim_{X_\pm \to \pm\infty} \lim_{D\to 0} \frac{2}{D^2} \lim_{\eta_\pm \to 0} (\eta_- \eta_+)^{\frac{1-D}{2}}$$

$$\times \int_{\eta(X_-)=\eta_-}^{\eta(X_+)=\eta_+} \mathcal{D}\eta(x) F[\hbar\eta^2(x)] \delta\left(\int_{X_-}^{X_+} \eta^2(x)dx - \beta\right) \eta(x_a)\eta(x_b) e^{-A_\Delta[\eta(x)]}$$

Summary

- We discussed local times of stochastic processes for generic Hamiltonian
- Calculated moments and distributions of local time
- In particular, for Lévy random walks
- For Gaussian path integrals derived local-time path-integral representation





P. Jizba and VZ, *Local-time representation of path integrals*, Phys. Rev. E **92**, 062137 (2015) [arXiv:1506.00888].

VZ, *Local time of Lévy random walks: a path integral approach*, [arXiv:1702.02488].