Czech Technical University in Prague

Faculty of Nuclear Sciences and Physical Engineering

BACHELOR'S THESIS

Applications of Supersymmetric Quantum Mechanics

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V Praze dne ....

Název práce:

Aplikace supersymetrické kvantové mechaniky

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Druh práce: Bakalářská práce

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Abstrakt:

Přestože supersymetrická kvantová mechanika (SUSY QM) byla původně vyvinuta pouze jako model pro testování supersymetrických polních teorií, brzy se ukázalo, že je tato oblast zajímavá i sama o sobě a že dokáže podstatným způsobem kvantovou mechaniku obohatit. Tato práce ukazuje základní myšlenky SUSY QM – faktorizace hamiltoniánu a definice jeho superpartnera – a dále se zaobírá některými důsledky a novými koncepty, které na těchto myšlenkách staví, jako jsou tvarově invariantní potenciály, zlomená supersymetrie a další. Závěr je zpestřen krátkým úvodem do problematiky geometrické (Berryho) fáze v naději, že se v budoucnu podaří při jejím studiu SUSY QM uplatnit.

Klíčová slova: supersymetrická kvantová mechanika, zlomená supersymetrie, tvarově invariantní potenciály, Berryho fáze

Title:

Applications of Supersymmetric Quantum Mechanics

Author: Václav Zatloukal

Abstract:

Although the supersymmetric quantum mechanics was originally developed as a model for testing supersymmetric field theories, it was soon evident that this domain was interesting of its own right and that it could considerably enrich the quantum mechanics itself. This thesis shows the basic ideas of SUSY QM – factorization of a Hamiltonian and definition of its superpartner – and discusses some of the consequences and new concepts which arose from these ideas, such as shape invariant potentials, broken supersymmetry and others. At the end of the thesis a short introduction into the issue of geometric (Berry) phase is given hoping that, in the future, SUSY QM will contribute to the development of this topic.

Keywords: supersymmetric quantum mechanics, broken supersymmetry, shape invariant potentials, Berry phase

Contents

Chapter 1

Introduction

Supersymmetry (SUSY) arose as a response to attempts by physicists to obtain a unified description of all basic interactions of nature. SUSY is, by definition, symmetry between fermions (i.e. matter or building blocks of matter) and bosons (i.e. field carriers). It may be noted here that so far there has been no experimental evidence of SUSY being realized in nature. Nevertheless, the ideas of SUSY have stimulated new approaches to other branches of physics like atomic, molecular, nuclear, statistical and condensed matter physics as well as nonrelativistic quantum mechanics.

Supersymmetric quantum mechanics (SUSY QM) was originally developed as a model for testing quantum field theory methods, but it was soon clear that this field was interesting in its own right. Gradually a whole technology was evolved based on SUSY to understand the solvable potential problems and even to discover new solvable potentials. The reason why certain potentials are exactly solvable can be understood in terms of few basic ideas which include supersymmetric partner potentials and shape invariance.

The Hamiltonian for SUSY QM is a 2×2 diagonal matrix Hamiltonian containing two separate Hamiltonians whose eigenvalues, eigenfunctions and scattering properties are, as we shall see, related because of existence of fermionic operators which commute with the matrix Hamiltonian.

At the end of this thesis I introduce the concept of geometric phase in QM (usually named after its very founder M.V.Berry). We shall see that during a cycling quantum evolution the phase change has two parts: the usual dynamical part and the geometric part, which is of our interest here, because it can have observable consequences.

Let me start with some fundamentals from one dimensional QM which we'll use later on in the text.

Chapter 2

Some Fundamental Theorems from One Dimensional Quantum Mechanics

In this chapter, we are concerned with the QM properties of a particle constrained to move along a straight line under the influence of a time-independent potential $V(x)$.

One dimensional quantum mechanical problems are not a pure mathematical construct. They appear, in fact, quite frequently when one studies higher dimensional problems. For instance, when we encounter a 2-D Hamiltonian of the form

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V_1(x) + V_2(y) = \underbrace{\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_1(x)\right)}_{H_1} + \underbrace{\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + V_2(y)\right)}_{H_2} ,$$

we can solve the time-independent Schrödinger equation

$$H\psi = E\psi \tag{2.1}$$

using the method of separation of variables, i.e. to write $\psi(x, y) = \psi_1(x)\psi_2(y)$. Then the equation (2.1) reads

$$(H_1 + H_2)\psi_1(x)\psi_2(y) = \psi_2(y)H_1\psi_1(x) + \psi_1(x)H_2\psi_2(y) = E\psi_1(x)\psi_2(y) .$$

Dividing by $\psi_1(x)\psi_2(y)$ separates the x and y dependence of the terms which hence must be constant:

$$\underbrace{\frac{H_1\psi_1(x)}{\psi_1(x)}}_{\equiv E_1} + \underbrace{\frac{H_2\psi_2(y)}{\psi_2(y)}}_{\equiv E_2} = E .$$

Thus the 2-D problem has been transformed into two 1-D problems: $H_1\psi_1(x) = E_1\psi_1(x)$ and $H_2\psi_2(y) = E_2\psi_2(y)$. Generalization to higher dimensions is straightforward.

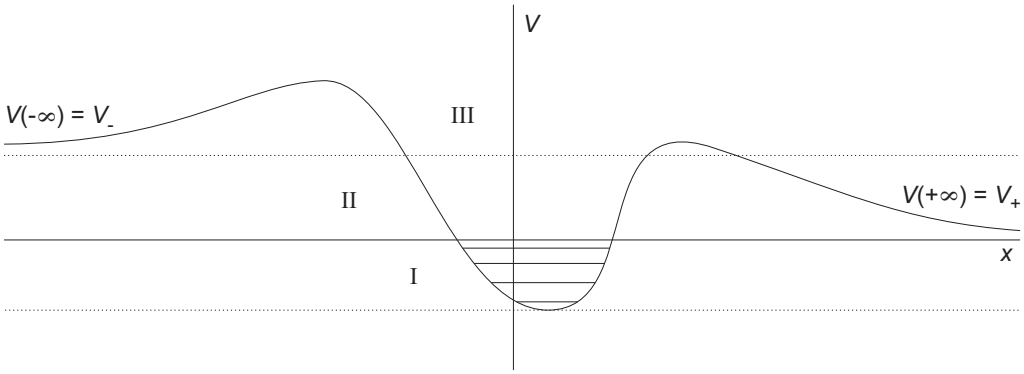

Figure 2.1: Generic shape of the potential $V(x)$ discussed throughout the chapter 2.

I will concentrate on the properties of a potential $V(x)$ with the generic shape shown on the Figure 2.1 and investigate the time-independent Schrödinger equation

$$H\psi = -\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi$$

or equivalently

$$\psi'' + \frac{2m}{\hbar^2}(E - V(x))\psi = 0 \ . \tag{2.2}$$

In the following two sections I state some general properties of eigenfunctions for both bound state and continuum situations. More rigorous derivation can be found in [3].

2.1 General Properties of Bound States

Let us first focus on the region I on the Figure 2.1 where $E < V_{\pm}$. The eigenfunction $\psi(x)$ must be square integrable, so it must approach zero (fast enough) as x goes to $\pm\infty$.

At $x \to -\infty$ our time-independent Schrödinger equation (2.2) takes the asymptotic form

$$\psi'' - \frac{2m}{\hbar^2}\underbrace{(V_- - E)}_{>0}\psi = 0$$

with the solution (that goes to 0 as $x \to -\infty$)

$$\psi_-(x) = C_- e^{k_- x} \ , \ k_- = \frac{\sqrt{2m(V_- - E)}}{\hbar} \ .$$

At $x \to +\infty$ we obtain analogously the solution

$$\psi_+(x) = C_+ e^{k_+ x} \ , \ k_+ = \frac{\sqrt{2m(V_+ - E)}}{\hbar} \ .$$

Inside of the potential well (where $E > V(x)$) we have the equation

$$\psi'' + \frac{2m}{\hbar^2}\underbrace{(E - V(x))}_{>0}\psi = 0 \ ,$$

whose solution is oscillatory. The requirement of continuity of ψ and ψ' on the potential well fixes E, which turns out[1] to be discrete. In addition, because ψ (being continuous) is bounded and goes exponentially to 0 at $x \to \pm\infty$, it is normalizable.

Since H is self-adjoint, its eigenvalues $E_0, E_1, ...$ are real and since we assume no additional degrees of freedom (such as spin) one can show that they are non-degenerate.[2] It also follows from self-adjointness of H that the eigenfunctions $\psi_0, \psi_1, ...$ are mutually orthogonal. Furthermore, thanks to non-degeneracy of the eigenvalues, the eigenfunctions can be chosen to be real.[3]

To get a little bit feeling for what the eigenfunctions look like I state the *oscillation theorem*, which holds for the bound states the energy located in the region I of the Fig.2.1: If the eigenvalues are ordered according to increasing energy, i.e. $E_0 < E_1 < E_2 < ...$, then the corresponding eigenfunctions are automatically ordered in the number of nodes, with the eigenfunction ψ_n having n nodes.[4] In addition, ψ_{n+1} has one node located between each pair of consecutive zeros in ψ_n (including the zeros at $x \to \pm\infty$).[5]

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[1] See [3], p.95

[2] Assume ψ_1 and ψ_2 are solutions of (2.2):

$$\psi_1'' + \frac{2m}{\hbar^2}(E - V(x))\psi_1 = 0 \ , \ \psi_2'' + \frac{2m}{\hbar^2}(E - V(x))\psi_2 = 0 \ .$$

This implies

$$\psi_1''\psi_2 = \psi_2''\psi_1$$
$$(\psi_1'\psi_2 - \psi_2'\psi_1)' = 0$$
$$\psi_1'\psi_2 - \psi_2'\psi_1 = K \ .$$

But $K = 0$, because $\psi_1(\pm\infty) = \psi_2(\pm\infty) = 0$. Hence

$$\frac{\psi_1'}{\psi_1} = \frac{\psi_2'}{\psi_2}$$

and so $\psi_1 = C\psi_2$, i.e. states ψ_1 and ψ_2 correspond to the same physical state.

[3] If ψ is a solution of (2.2) then both $\Re\psi$ and $\Im\psi$ must be solutions as well. From non-degeneracy of the spectrum it follows that $\Im\psi = C\Re\psi$ and

$$\psi = \Re\psi + i\Im\psi = (1 + iC)\Re\psi \ .$$

[4] For details see [3], p.95

[5] Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ be the two consecutive zeros of ψ_n, $n = 0, 1, ...$ From

$$\psi_n'' + \frac{2m}{\hbar^2}(E_n - V(x))\psi_n = 0 \ , \ \psi_{n+1}'' + \frac{2m}{\hbar^2}(E_{n+1} - V(x))\psi_{n+1} = 0$$

we get (multiplying the first equality by ψ_{n+1}, the second by ψ_n and subtracting them)

$$(\psi_n'\psi_{n+1} - \psi_{n+1}'\psi_n)' = \psi_n''\psi_{n+1} - \psi_{n+1}''\psi_n = \frac{2m}{\hbar^2}(E_{n+1} - E_n)\psi_n\psi_{n+1} \ .$$

Integrating from a to b and using $\psi_n(a) = \psi_n(b) = 0$ yields

$$[\psi_n'\psi_{n+1}]_a^b = \frac{2m}{\hbar^2}(\underbrace{E_{n+1} - E_n}_{>0})\int_a^b \psi_n(x)\psi_{n+1}(x)dx \ .$$

2.2 General Properties of Continuum States and Scattering

See again Fig.2.1. In the region II, i.e. for $V_+ < E < V_-$, the spectrum is continuous and non-degenerate.

For $x \to -\infty$ the equation (2.2) takes form

$$\psi'' - \frac{2m}{\hbar^2} \underbrace{(V_- - E)}_{>0} \psi = 0$$

with just one bounded solution (which tends to 0 at $x \to -\infty$)

$$\psi_-(x) = C_- e^{k_- x} \ , \ k_- = \frac{\sqrt{2m(V_- - E)}}{\hbar} \ .$$

On the other hand, in asymptotic neighborhood of $+\infty$ the Schrödinger equation

$$\psi'' + \frac{2m}{\hbar^2} \underbrace{(E - V_+)}_{>0} \psi = 0$$

has oscillatory solutions in the form of *stationary plane waves*

$$\psi(x) = A\cos(kx + \delta) \ , \ k = \frac{\sqrt{2m(E - V_+)}}{\hbar} \ .$$

A and δ are constants such that ψ, ψ' are continuous.

In the region III of Fig.2.1 (where $E > V_-$) the spectrum is continuous and twice degenerate. Since $E > V_- > V_+$ the asymptotic states at $\pm\infty$ are described by plane waves

$$\psi_\pm(x) = A_\pm e^{ik_\pm x} + B_\pm e^{-ik_\pm x}$$

The degeneracy follows from the fact that there are two independent solutions, both bounded at $\pm\infty$. (Whereas in region II one of the solutions became infinity and therefore had to be rejected.)

Spectrum is continuous as there is no condition fixing E.

In the region III we may construct wave functions which are partly transmitted and partly reflected by $V(x)$:

$$\psi(x) = \begin{cases} e^{ik_- x} + R(k_-)e^{-ik_- x} & x \to -\infty \\ T(k_-)e^{ik_+ x} & x \to +\infty \end{cases} \ ,$$

where $R(k_-)$, $T(k_-)$[6] are called reflection and transmission amplitudes (or coefficients), respectively. Conservation of probability guarantees that

$$|T(k_-)|^2 + |R(k_-)|^2 = 1 \ .$$

In the interval (a, b) ψ_n keeps the same sign (say $\psi_n > 0$). Thus $\psi_n'(a) \geq 0$ and $\psi_n'(b) \leq 0$. Hence ψ_{n+1} certainly changes sign in (a, b). If not (say $\psi_{n+1} > 0$), then the RHS would be > 0, but the LHS would be ≤ 0. Because ψ_{n+1} has just one more node than ψ_n, it obviously cannot have more than one node between each pair of consecutive zeros of ψ_n.

[6] R and T depend on the energy $E = \frac{\hbar^2 k_-^2}{2m} + V_- = \frac{\hbar^2 k_+^2}{2m} + V_+$ of the wave function ψ. I use this "k_- convention" although I could have equally well written $R(k_+)$, $T(k_+)$ or $R(E)$, $T(E)$.

Chapter 3

Supersymmetric Quantum Mechanics (SUSY QM)

The fact that certain Hamiltonians can be factorized in the way that is shown at the beginning of this chapter has many consequences. Later on we will discuss some of them to illustrate that SUSY QM is an interesting as well as a useful concept.

3.1 Factorization of a General Hamiltonian

Let us have a single particle Hamiltonian

$$H_1 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x)$$

with some potential $V_1(x)$ and suppose the ground state wave function $\psi_0(x)$ is specified. It is quite an interesting observation that ψ_0 also specifies the potential $V_1(x)$ (up to a constant). Without loss of generality, we can choose the ground state energy $E_0^{(1)}$ of H_1 to be zero.[1]

Inserting ψ_0 into the time-independent Schrödinger equation then yields

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} + V_1(x)\psi_0(x) = 0 \ ,$$

which can be rewritten (using the fact that ψ_0 is nodeless) as

$$V_1(x) = \frac{\hbar^2}{2m}\frac{\psi_0''(x)}{\psi_0(x)} \ . \tag{3.1}$$

Notice that rescaling $\psi_0(x) \to e^{i\varphi}\psi_0(x)$ doesn't give different $V_1(x)$.

Now I factorize our Hamiltonian as follows:

$$H_1 = A^\dagger A \ ,$$

[1] If it wasn't, we would just work with shifted Hamiltonian $H_1 - E_0^{(1)}$.

where

$$A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) \ , \ A^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x) \ . \tag{3.2}$$

To find $W(x)$, we expand $A^\dagger A$:

$$H_1 = A^\dagger A = \left(-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right)\left(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x) \ .$$

This allows us to identify

$$V_1(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x) \ , \tag{3.3}$$

which is the Riccati equation for $W(x)$. One can either solve the equation (3.3) by standard manners ([10]) or use a little trick.

We know that ψ_0 must obey $H_1\psi_0 = A^\dagger A\psi_0 = 0$, which is satisfied automatically if $A\psi_0 = 0$. This requirement means (in view of the relation (3.2))

$$0 = A\psi_0 = \left(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right)\psi_0$$

$$\Rightarrow W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'(x)}{\psi_0(x)} \ . \tag{3.4}$$

It is obvious that this $W(x)$ really satisfies (3.3), once we take into account the expression for $V_1(x)$ (3.1). The expression (3.4) is not the only solution of (3.3), however, the other solutions are more complicated and we shall not use them. The quantity $W(x)$ defined by (3.4) is referred to as the *superpotential* in SUSY QM literature.

The next step in constructing the SUSY theory is to define the operator

$$H_2 := AA^\dagger = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x) \ .$$

H_2 is in fact a Hamiltonian corresponding to a new potential $V_2(x)$:

$$H_2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x) \ , \ V_2(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x) \ . \tag{3.5}$$

The potentials $V_1(x)$ and $V_2(x)$ are known as supersymmetric partner (or superpartner) potentials. Similarly, H_1 and H_2 are called superpartner Hamiltonians.

3.1.1 Relation of the Eigenstates of Two Superpartner Hamiltonians

As we shall see in this paragraph, the eigenstates (i.e. the energy eigenvalues and the eigenfunctions) of H_1 and H_2 are related.

Let us take an arbitrary eigenstate $\psi_n^{(1)}$ of H_1. Then the following holds for all $n \geq 0$:

$$H_1\psi_n^{(1)} = A^\dagger A\psi_n^{(1)} = E_n^{(1)}\psi_n^{(1)}$$

implies

$$H_2(A\psi_n^{(1)}) = AA^\dagger A\psi_n^{(1)} = E_n^{(1)}(A\psi_n^{(1)}) \ . \tag{3.6}$$

Conversely, if we take an arbitrary eigenstate $\psi_n^{(2)}$ of H_2, then

$$H_2\psi_n^{(2)} = AA^\dagger\psi_n^{(2)} = E_n^{(2)}\psi_n^{(2)}$$

implies

$$H_1(A^\dagger\psi_n^{(2)}) = A^\dagger AA^\dagger\psi_n^{(2)} = E_n^{(2)}(A^\dagger\psi_n^{(2)}) \ . \tag{3.7}$$

Since both H_1 and H_2 are positive operators[2] their spectra are subsets of $[0, +\infty)$.

Now I show that zero is not an eigenvalue of H_2. For this purpose let's realize that there is no non-zero square integrable function ψ such that $A^\dagger\psi = 0$.[3] Suppose now that there exists a (non-zero) wave function $\tilde{\psi}$ such that $H_2\tilde{\psi} = 0$. Then

$$0 \neq \|A^\dagger\tilde{\psi}\|^2 = \langle\tilde{\psi} \mid H_2 \mid \tilde{\psi}\rangle = 0$$

and so zero is not in the spectrum of H_2.

From (3.6) we can see that $E_n^{(1)}$ is also an eigenvalue of H_2 if $A\psi_n^{(1)} \neq 0$, which is violated (i.e. $A\psi_n^{(1)} = 0$) only for $n = 0$.[4] Similarly, from (3.7) it follows that $E_n^{(2)}$ is also an eigenvalue of H_1 if $A^\dagger\psi_n^{(2)} \neq 0$, but this holds for all (square integrable) eigenfunctions as shown in the footnote 3.

We conclude with the statement that the eigenvalues of H_1 and H_2 (ordered increasingly) are related by $(n = 0, 1, ...)$

$$E_n^{(2)} = E_{n+1}^{(1)} \ , \ E_0^{(1)} = 0 \ . \tag{3.8}$$

Further, as seen from (3.6) and (3.7) and using the spectra relation (3.8), the eigenfunctions of H_1 and H_2 are proportional to each other:

$$\psi_{n+1}^{(1)} \propto A^\dagger\psi_n^{(2)} \ , \ \psi_n^{(2)} \propto A\psi_{n+1}^{(1)} \ ,$$

[2] $H_1 = A^\dagger A \Rightarrow \langle\psi \mid H_1 \mid \psi\rangle = \|A\psi\|^2$, $H_2 = AA^\dagger \Rightarrow \langle\psi \mid H_2 \mid \psi\rangle = \|A^\dagger\psi\|^2$

[3] Because the wave function $\psi_0^{(1)}$ satisfies

$$A\psi_0^{(1)} = \left(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right)\psi_0^{(1)} = 0 \ ,$$

we can write explicitly $\psi_0^{(1)}(x) = C_1 \exp\left(-\frac{\sqrt{2m}}{\hbar}\int_a^x W(y)dy\right)$. Our condition for ψ

$$A^\dagger\psi = \left(-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right)\psi = 0$$

yields $\psi(x) = C_2 \exp\left(\frac{\sqrt{2m}}{\hbar}\int_a^x W(y)dy\right)$. Thus $\psi_0^{(1)}\psi = C_1C_2 \neq 0$. Now it is obvious that since $\psi_0^{(1)}$ is square integrable (and hence vanishes at $\pm\infty$), ψ cannot be. I shall use similar argument in the section 3.2 talking about broken supersymmetry.

[4] The equation $A\psi = \left(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right)\psi = 0$ has only one solution (up to a multiplicative constant) and we know that $A\psi_0 = 0$ from the definition of $W(x)$ (3.4).

for $n = 0, 1, \ldots$. To get the proportionality constants c_n, we use the normalization condition. Assume that $\{\psi_n^{(1)}\}_n$ are normalized. We want $\{\psi_n^{(2)}\}_n$ to be normalized as well. We get

$$1 = \langle \psi_{n+1}^{(1)} \mid \psi_{n+1}^{(1)} \rangle = \langle c_n A^\dagger \psi_n^{(2)} \mid c_n A^\dagger \psi_n^{(2)} \rangle = |c_n|^2 \langle \psi_n^{(2)} \mid \underbrace{A A^\dagger}_{H_2} \mid \psi_n^{(2)} \rangle = |c_n|^2 E_n^{(2)}$$

and our choice of c_n's as positive real numbers leads us to the relation between the superpartner eigenfunctions

$$\psi_{n+1}^{(1)} = \frac{1}{\sqrt{E_n^{(2)}}} A^\dagger \psi_n^{(2)} \ , \tag{3.9}$$

for $n = 0, 1, \ldots$. Similarly we obtain

$$\psi_n^{(2)} = \frac{1}{\sqrt{E_{n+1}^{(1)}}} A \psi_{n+1}^{(1)} \ , \tag{3.10}$$

where again $n = 0, 1, \ldots$.

So we can say that knowing all the eigenfunctions of H_1, we can determine the eigenfunctions of H_2 using the operator A and, vice versa, using $A^\dagger$ we can reconstruct all the eigenfunctions of H_1 from those of H_2 (except for the ground state). The correspondence is illustrated in Fig.3.1

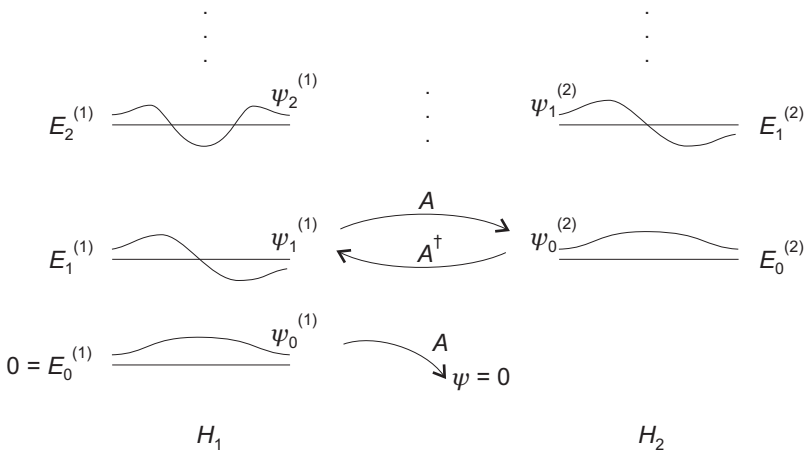

Figure 3.1: Energy levels of two (unbroken) supersymmetric partner Hamiltonians H_1 and H_2. The energy levels are degenerate except that H_1 has one extra state at zero energy. The action of the operators A and $A^\dagger$ is displayed and the number of nodes of the eigenfunctions of H_1 and H_2 is indicated.

3.1.2 Example: Infinite Square Well and Its Superpartner

We have seen that if there is an exactly solvable potential with at least one bound state, then we can always construct its SUSY partner potential and it is also exactly solvable.

Let us look at the most elementary potential, namely the infinite square well, and let us determine its SUSY partner potential.

Consider a particle of mass m in an infinite square well potential of width L:

$$V(x) = \begin{cases} 0 & x \in [0, L] \\ +\infty & x \in \mathbb{R} \setminus [0, L] \ . \end{cases}$$

The normalized ground state wave function is known to be (see [3], p.76)

$$\psi_0^{(1)}(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} \ , \ x \in [0, L] \ ,$$

and the ground state energy is

$$E_0 = \frac{\hbar^2 \pi^2}{2mL^2} \ .$$

Subtracting off the ground state energy so that the Hamiltonian can be factorized, we have for $H_1 := H - E_0$ that the energy eigenvalues are

$$E_n^{(1)} = E_n - E_0 = \frac{(n+1)^2 \pi^2 \hbar^2}{2mL^2} - \frac{\pi^2 \hbar^2}{2mL^2} = \frac{n(n+2)\pi^2 \hbar^2}{2mL^2} \ , \ n = 0, 1, ... \ .$$

Since H and H_1 have the same eigenfunctions we can write down the normalized eigenfunctions of H_1:

$$\psi_n^{(1)} = \sqrt{\frac{2}{L}} \sin \frac{(n+1)\pi x}{L} \ , \ x \in [0, L] \ , \ n \geq 0 \ .$$

The superpotential for this problem is readily obtained using (3.4) as

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\left(\sqrt{\frac{2}{L}} \sin \frac{\pi x}{L}\right)'}{\sqrt{\frac{2}{L}} \sin \frac{\pi x}{L}} = -\frac{\hbar}{\sqrt{2m}} \frac{\pi}{L} \cot \frac{\pi x}{L}$$

and hence the supersymmetric partner potential V_2 is (substituting $W(x)$ into (3.5))

$$V_2(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x) = \frac{\hbar^2 \pi^2}{2mL^2} \left(\frac{\cos^2 \frac{\pi x}{L}}{\sin^2 \frac{\pi x}{L}} + \frac{1}{\sin^2 \frac{\pi x}{L}} \right) = \frac{\hbar^2 \pi^2}{2mL^2} \left(\frac{2}{\sin^2 \frac{\pi x}{L}} - 1 \right) \ .$$

Figure 3.2 shows $V_1(x)$, $V_2(x)$, $W(x)$ and $\psi_0^{(1)}(x)$ of our supersymmetric problem.

The wave functions for H_2 are obtained by applying the operator $A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x)$ to the wave functions of H_1. In particular we find that the normalized ground and first excited state wave functions of H_2 are

$$\psi_0^{(2)} = \frac{A\psi_1^{(1)}}{\sqrt{E_1^{(1)}}} = \frac{1}{\sqrt{\frac{3\pi^2\hbar^2}{2mL^2}}} \sqrt{\frac{2}{L}} \left(\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} - \frac{\hbar}{\sqrt{2m}} \frac{\pi}{L} \cot \frac{\pi x}{L} \right) \sin \frac{2\pi x}{L} = -2\sqrt{\frac{2}{3L}} \sin^2 \frac{\pi x}{L} \ ,$$

$$\psi_1^{(2)} = \frac{A\psi_2^{(1)}}{\sqrt{E_2^{(1)}}} = \frac{1}{\sqrt{\frac{8\pi^2\hbar^2}{2mL^2}}} \sqrt{\frac{2}{L}} \left(\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} - \frac{\hbar}{\sqrt{2m}} \frac{\pi}{L} \cot \frac{\pi x}{L} \right) \sin \frac{3\pi x}{L} = -\frac{2}{\sqrt{L}} \sin \frac{\pi x}{L} \sin \frac{2\pi x}{L} \ .$$

The other eigenfunctions of H_2 can be obtained analogously.

Thus I have shown using SUSY that two rather different potentials corresponding to H_1 and H_2 have exactly the same spectra except for the fact that H_2 has one fewer bound state.

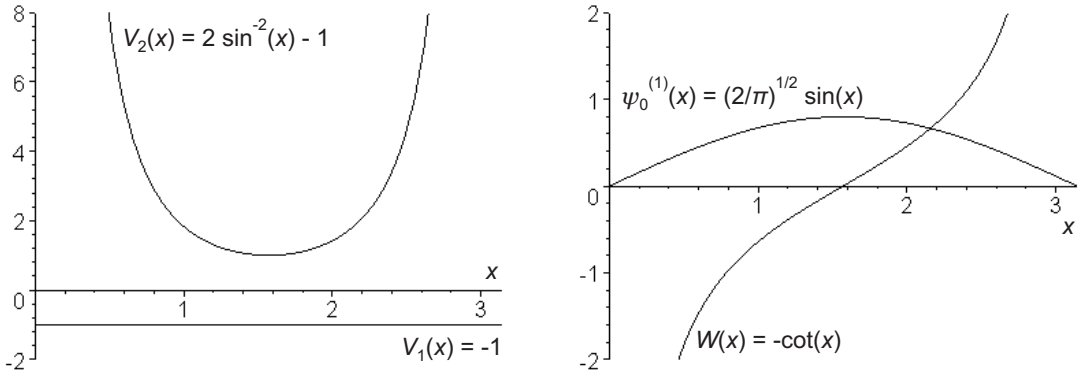

Figure 3.2: On the left the infinite square well potential $V_1(x) = -1$ and its partner potential $V_2(x) = \frac{2}{\sin^2 x} - 1$. For convenience we put the well width $L = \pi$ and $\hbar = 2m = 1$. On the right the ground state eigenfunction of H_1 $\psi_0^{(1)}(x)$ and the superpotential for our problem $W(x)$.

3.1.3 SUSY Algebra

The energy matching of H_1 and H_2 has a deeper algebraic reason. Let us construct a SUSY matrix Hamiltonian of the form

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} ,$$

which contains both H_1 and H_2, and consider operators (so called supercharges)

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} , \; Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix} .$$

It is an easy observation that Q and $Q^\dagger$ in conjunction with H form a closed graded algebra which contains both bosonic (H) and fermionic ($Q, Q^\dagger$) operators with commutation and anti-commutation relations:

$$[H, Q] = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ H_2 A - A H_1 & 0 \end{pmatrix} = 0 , \tag{3.11}$$

$$[H, Q^\dagger] = -[H, Q]^\dagger = 0 , \tag{3.12}$$

$$\{Q, Q^\dagger\} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & A A^\dagger \end{pmatrix} = H , \tag{3.13}$$

$$\{Q, Q\} = 2Q^2 = 0 \ , \tag{3.14}$$

$$\{Q^\dagger, Q^\dagger\} = 2{Q^\dagger}^2 = 0 \ . \tag{3.15}$$

The graded algebra (3.11) - (3.15) was directly motivated by the supersymmetric algebra in QFT. (See, e.g. [5].)

Suppose now that Ψ is an eigenfunction of H corresponding to an eigenvalue E, i.e. $H\Psi = E\Psi$. From the commutation relations (3.11), (3.12) it follows that

$$QH\Psi = H(Q\Psi) = E(Q\Psi) \ , \tag{3.16}$$

$$Q^\dagger H\Psi = H(Q^\dagger\Psi) = E(Q^\dagger\Psi) \ .$$

If we now take Ψ in the form

$$\Psi = \begin{pmatrix} \psi^{(1)} \\ 0 \end{pmatrix} \ ,$$

where $H_1\psi^{(1)} = E\psi^{(1)}$, then

$$H\Psi = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \begin{pmatrix} \psi^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} H_1\psi^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} E\psi^{(1)} \\ 0 \end{pmatrix} = E\Psi \ .$$

Hence

$$Q\Psi = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} \psi^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ A\psi^{(1)} \end{pmatrix}$$

must satisfy (3.16) which now reads

$$\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \begin{pmatrix} 0 \\ A\psi^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ H_2(A\psi^{(1)}) \end{pmatrix} = \begin{pmatrix} 0 \\ E(A\psi^{(1)}) \end{pmatrix} \ . \tag{3.17}$$

Analogously we can obtain

$$\begin{pmatrix} H_1(A^\dagger\psi^{(2)}) \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{E}(A^\dagger\psi^{(2)}) \\ 0 \end{pmatrix} \ , \tag{3.18}$$

where $H_2\psi^{(2)} = \tilde{E}\psi^{(2)}$. Notice that (3.17) and (3.18) are in fact the equalities (3.6) and (3.7) from section 3.1.1 which enabled us to relate the eigenstates of H_1 and H_2.

3.1.4 Relation of the Reflection and Transmission Coefficients of SUSY Partners

Another important consequence of SUSY QM is that it allows to relate the reflection and transmission coefficients in situations when the two partner potentials have continuous spectra.

At the beginning of section 3.1 I introduced supersymmetry of two potentials using the ground state wave function $\psi_0^{(1)}$ of H_1. In fact we don't need any bound state wave function to construct the SUSY partner Hamiltonian H_2 and obtain relation between H_1 and H_2. We may just use (3.2) – the definition of A and $A^\dagger$ – and for a given $H_1 = A^\dagger A$ define $H_2 := AA^\dagger$ with $W(x)$ being some solution of the Riccati equation (3.3).

In order for scattering to take place in both of the partner potentials, it is necessary that $V_{1,2}$ are finite as $x \to -\infty$ or as $x \to +\infty$ or both. If $V_{1,2}$ is unbounded both for $x \to \pm\infty$, then we do not have free particle, because the wave function disappears at $x \to \pm\infty$ exponentially. Transmission and reflection coefficients are only defined for particles which have well defined plane wave properties at $x \to -\infty$ or $x \to +\infty$ (or both).

Let us presume that there exist finite limits $\lim_{x\to\pm\infty} W(x), \lim_{x\to\pm\infty} W'(x)$ and let us define

$$W_\pm := W(x \to \pm\infty) \ .$$

Then $W'(x \to \pm\infty)$ must be zero and it follows from (3.3) and (3.5) that

$$V_1(x \to \pm\infty) = V_2(x \to \pm\infty) = W^2(x \to \pm\infty) \ .$$

See Figure 3.3 in section 3.1.5 for example.

Let us consider, for definiteness, an incident plane wave $e^{ik_- x}$ of energy E coming from $-\infty$. As a result of scattering from the potentials $V_{1,2}(x)$ one would obtain transmitted waves $T_{1,2}(k_-)e^{ik_+ x}$ and reflected waves $R_{1,2}(k_-)e^{-ik_- x}$. The boundary conditions are

$$\psi_E^{(1,2)}(x \to -\infty) = e^{ik_- x} + R_{1,2}(k_-)e^{-ik_- x} \ , \tag{3.19}$$

$$\psi_E^{(1,2)}(x \to +\infty) = T_{1,2}(k_-)e^{ik_+ x} \ , \tag{3.20}$$

where k_- and k_+ are given by

$$k_\pm = \frac{\sqrt{2m(E - W_\pm^2)}}{\hbar} \ .$$

SUSY connects continuum wave functions of H_1 and H_2 having the same energy analogously to what happens in the discrete spectrum – if $\psi_E^{(1)}$ satisfies $H_1\psi_E^{(1)} = E\psi_E^{(1)}$, then

$$H_2(A\psi_E^{(1)}) = AH_1\psi_E^{(1)} = E(A\psi_E^{(1)}) \Rightarrow \psi_E^{(2)} = NA\psi_E^{(1)} \ ,$$

where N is a normalization constant. Using the definition of the operator A (3.2) and expressions (3.19), (3.20) for $\psi_E^{(1)}$ we find that in the asymptotic region $x \to -\infty$

$$\psi_E^{(2)}(x \to -\infty) = NA\psi_E^{(1)}(x \to -\infty) = N\left(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W_-\right)\left(e^{ik_- x} + R_1(k_-)e^{-ik_- x}\right)$$

$$= N\left(\frac{i\hbar k_-}{\sqrt{2m}} + W_-\right)e^{ik_- x} + N\left(\frac{-i\hbar k_-}{\sqrt{2m}} + W_-\right)R_1(k_-)e^{-ik_- x} \ .$$

Equating the latter with (3.19) (terms with the same exponent) gives us

$$N\left(\frac{i\hbar k_-}{\sqrt{2m}}+W_-\right)=1\ ,\ N\left(\frac{-i\hbar k_-}{\sqrt{2m}}+W_-\right)R_1(k_-)=R_2(k_-)\ . \tag{3.21}$$

Let us do the same thing at $+\infty$:

$$\psi_E^{(2)}(x\to+\infty)=NA\psi_E^{(1)}(x\to+\infty)=N(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}+W_+)T_1(k_-)e^{ik_+x}$$

$$=N(\frac{i\hbar k_+}{\sqrt{2m}}+W_+)T_1(k_-)e^{ik_+x}\ .$$

Hence

$$N(\frac{i\hbar k_+}{\sqrt{2m}}+W_+)T_1(k_-)=T_2(k_-)\ . \tag{3.22}$$

Finally, let us take together relations (3.21) and (3.22) and find

$$R_1(k_-)=\frac{\frac{i\hbar k_-}{\sqrt{2m}}+W_-}{\frac{-i\hbar k_-}{\sqrt{2m}}+W_-}R_2(k_-)\ ,\ T_1(k_-)=\frac{\frac{i\hbar k_-}{\sqrt{2m}}+W_-}{\frac{i\hbar k_+}{\sqrt{2m}}+W_+}T_2(k_-)\ . \tag{3.23}$$

It is now an easy consequence that the reflection and transmission probabilities of two partner potentials are identical, i.e.

$$|R_1|^2=|R_2|^2 \text{ and } |T_1|^2=|T_2|^2\ .^5 \tag{3.24}$$

3.1.5 Example: A Reflectionless Potential

Clearly, a constant potential $V_1(x)\equiv C$ (i.e. a free particle) is reflectionless ($|R_1|=0$). It follows from (3.24) that its SUSY partner potential is also reflectionless. Let's find some nontrivial partner potential to the free particle potential.

Let's first take the equation (3.3), which reads

$$V_1(x)\equiv C=W^2(x)-\frac{\hbar}{\sqrt{2m}}W'(x)\ , \tag{3.25}$$

and solve it using the anzatz $W(x)=-\frac{\hbar}{\sqrt{2m}}\frac{u'(x)}{u(x)}$. We get a second order linear differential equation for new unknown function $u(x)$

$$\frac{2mC}{\hbar^2}u(x)=u''(x)$$

with general solution

$$u(x)=D_1\exp\left(\frac{\sqrt{2mC}}{\hbar}x\right)+D_2\exp\left(-\frac{\sqrt{2mC}}{\hbar}x\right)\ ,\ D_1,D_2\in\mathbb{R}\ .$$

[5] Recall that $k_\pm=\frac{\sqrt{2m(E-W_\pm^2)}}{\hbar}$. Thus $\frac{\hbar^2k_+^2}{2m}+W_+^2=\frac{\hbar^2k_-^2}{2m}+W_-^2=E$.

The partner potential is given by (3.5), i.e. (considering (3.25)) as

$$V_2(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x) = 2W^2(x) - C = 2\frac{\hbar^2}{2m}\frac{u'^2(x)}{u^2(x)} - C \ .$$

For the simplicity sake I choose $D_1 = D_2 = \frac{1}{2}$ and suppose that $C > 0$. Then $u(x) = \cosh \frac{\sqrt{2mC}}{\hbar} x$ and we get

$$V_2(x) = 2\frac{\hbar^2}{2m}\frac{2mC}{\hbar^2}\frac{\sinh^2 \frac{\sqrt{2mC}}{\hbar} x}{\cosh^2 \frac{\sqrt{2mC}}{\hbar} x} - C = 2C\left(1 - \frac{1}{\cosh^2 \frac{\sqrt{2mC}}{\hbar} x}\right) - C = C - \frac{2C}{\cosh^2 \frac{\sqrt{2mC}}{\hbar} x} \ .$$

We have obtained a nontrivial reflectionless potential or one may say we have proven reflectionlessness of the potential $V_2(x) = C - 2C \cosh^{-2} \frac{\sqrt{2mC}}{\hbar} x$. The functions $V_1(x)$, $V_2(x)$ and $W(x)$ are plotted in the Figure 3.3.

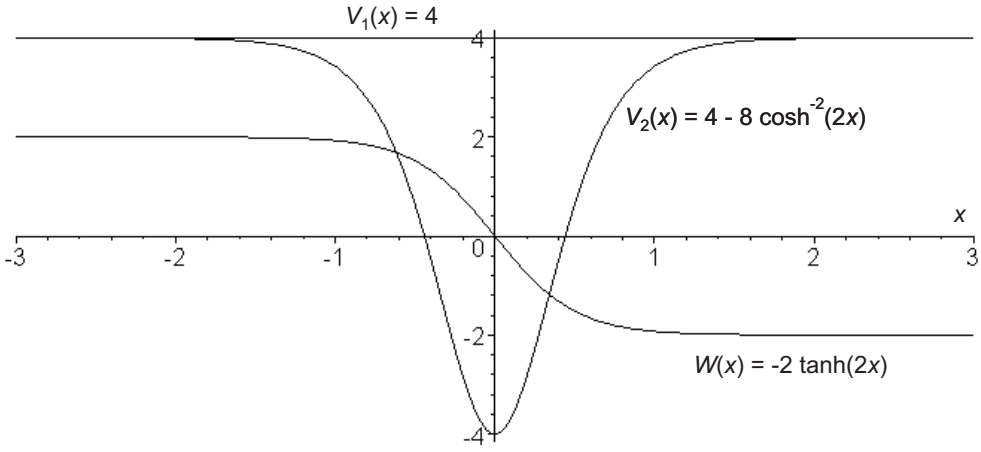

Figure 3.3: The free particle potential $V_1(x)$, its superpartner potential $V_2(x)$ and the superpotential $W(x)$ plotted for $C = 4$, $\hbar = 2m = 1$. Since V_1 is clearly reflectionless, V_2 is also a reflectionless potential.

3.2 Broken Supersymmetry

A symmetry of the Hamiltonian (or Lagrangian) is told to be spontaneously broken if the lowest energy solution does not respect that symmetry. Consider, for example, ferromagnet, where the Hamiltonian has rotational symmetry, but we know that at $T \to 0$ it picks up magnetization in one direction.

Another example from everyday life may be a straw. If you stress a straw, it bends. Even though your description prefers no direction, it must bend somewhere, so it spontaneously picks up one direction and bends. By choosing one of true vacua the system spontaneously violates its rotational symmetry (see Fig.3.4).

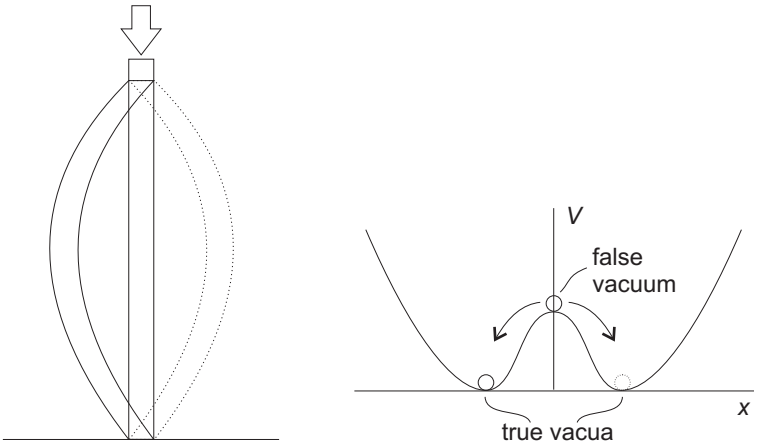

Figure 3.4: A stressed straw spontaneously brakes rotational symmetry choosing one of the true vacua (the lowest energy states).

So far we have seen that when the ground state wave function of H_1 is known, then we can factorize the Hamiltonian and find the SUSY partner Hamiltonian H_2. Now let us consider the converse problem. Suppose we are given a superpotential $W(x)$ and construct the matrix Hamiltonian H (as we did in section 3.1.3). There are two candidates for the zero energy ground state wave function of H:

$$\begin{pmatrix} \psi_0^{(1)} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \psi_0^{(2)} \end{pmatrix} ,$$

where $\psi_0^{(1)}$ and $\psi_0^{(2)}$ are the zero energy ground states of H_1 and H_2, respectively, and can be obtained from:[6]

$$A\psi_0^{(1)}(x) = 0 \Rightarrow \psi_0^{(1)}(x) = C_1 \exp\big(-\frac{\sqrt{2m}}{\hbar} \int_a^x W(y)dy \big) ,$$

$$A^\dagger \psi_0^{(2)}(x) = 0 \Rightarrow \psi_0^{(2)}(x) = C_2 \exp\big(\frac{\sqrt{2m}}{\hbar} \int_a^x W(y)dy \big) .$$

Because $\psi_0^{(1)}\psi_0^{(2)} = C_1C_2 \neq 0$, $\psi_0^{(1)}$ and $\psi_0^{(2)}$ cannot both vanish at $\pm\infty$ and therefore cannot be both square integrable at the same time. By convention, we shall always choose W in such a way that amongst H_1, H_2 only H_1 (if at all) will have a normalizable zero energy ground state eigenfunction. This is ensured by choosing W such that $W(x)$ is positive (negative) for large positive (negative) x.

[6] If $H_1\psi_0^{(1)} = 0$, then

$$0 = \langle \psi_0^{(1)} | H_1 | \psi_0^{(1)} \rangle = \langle \psi_0^{(1)} | A^\dagger A | \psi_0^{(1)} \rangle = \| A\psi_0^{(1)} \|^2$$

implies $A\psi_0^{(1)} = 0$. Analogously for $\psi_0^{(2)}$.

Let us denote the ground state of H by a two component vector $|0\rangle$. Supersymmetry in QM is said to be an unbroken symmetry (or exact SUSY) if

$$Q|0\rangle = Q^\dagger|0\rangle = 0 \ , \tag{3.26}$$

where Q and $Q^\dagger$ have been defined in section 3.1.3. It follows from (3.13) that in this case $H|0\rangle = 0$. Thus $|0\rangle$ can be written as

$$|0\rangle = \begin{pmatrix} \psi_0^{(1)} \\ 0 \end{pmatrix} \ .$$

In all the cases we have encountered so far the SUSY was indeed unbroken.

If neither $\psi_0^{(1)}$ nor $\psi_0^{(2)}$ is normalizable then H does not have a zero energy eigenstate and SUSY is (spontaneously) broken.[7] Denote $\varphi_0^{(1)}$ the ground state of H_1, i.e. $H_1\varphi_0^{(1)} = E_0^{(1)}\varphi_0^{(1)} \neq 0$. The operator A (defined by (3.2)) no longer annihilates $\varphi_0^{(1)}$. Thus (3.6) shows that $A\varphi_0^{(1)} \neq 0$ is an eigenstate of H_2 corresponding to the eigenvalue $E_0^{(1)}$, which is in turn the ground state energy of H_2. The relations between the eigenstates of H_1 and H_2 that one now obtains are ($n = 0, 1, ...$):

$$E_n^{(2)} = E_n^{(1)} > 0 \ ,$$

$$\varphi_n^{(1)} = \frac{1}{\sqrt{E_n^{(2)}}} A^\dagger \varphi_n^{(2)} \ ,$$

$$\varphi_n^{(2)} = \frac{1}{\sqrt{E_n^{(1)}}} A\varphi_n^{(1)} \ .$$

The situation is sketched in the Figure 3.5.

Consider, for example, superpotentials of the form

$$W(x) = gx^n \ , \ g > 0 \ , \ n \in \mathbb{N} \ .$$

Then for n odd one always has a normalizable ground state wave function (SUSY is unbroken). However for the case of n even, there is no candidate matrix ground state wave function that is normalizable (SUSY is broken).

The breaking of SUSY can be described by a topological quantum number called the Witten index (see [1], p.26).

3.3 Shape Invariance and Solvable Potentials

In 1983, the concept of a shape invariant potential (SIP) within the structure of SUSY QM was introduced by Gendenshtein. The definition presented was as follows: a potential is said to be shape invariant if its SUSY partner potential has the same spatial dependence as the

[7] Since $H|0\rangle \neq 0$, where $|0\rangle$ is the ground state of H, the condition of unbroken SUSY (3.26) does not hold.

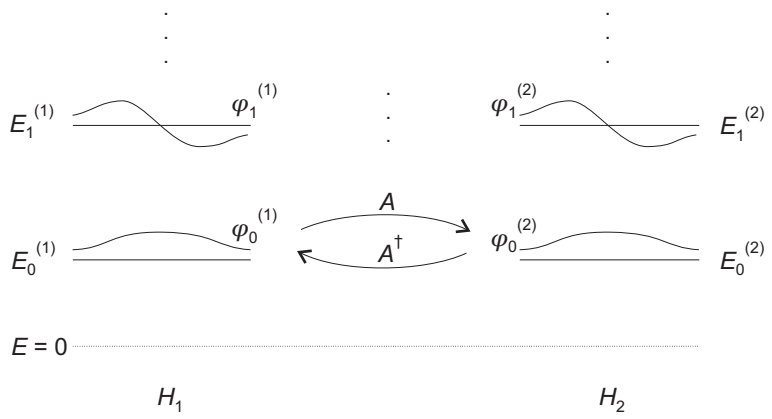

Figure 3.5: Energy levels of two SUSY partner Hamiltonians H_1 and H_2 in the case of broken SUSY. All the energy levels are degenerate. The action of the operators A and $A^\dagger$ is displayed and the number of nodes of the eigenfunctions of H_1 and H_2 is indicated. To be compared with Figure 3.1 which represents the case of unbroken SUSY.

original potential with possibly altered parameters. It was readily shown that for any SIP, the energy eigenvalue spectrum can be obtained algebraically. Much later, a list of SIPs was given and it was shown that the energy eigenfunctions as well as the scattering matrix could also be obtained algebraically for these potentials.

3.3.1 The Shape Invariance Condition

I start this section with one rather innocent example.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega x^2 \tag{3.27}$$

is the harmonic oscillator Hamiltonian. It's ground state eigenvalue and (unnormalized) ground state eigenfunction are known to be ([3], p.80) $E_0 = \frac{\hbar\omega}{2}$ and $\psi_0(x) = \exp(-\frac{m\omega}{2\hbar}x^2)$. To get the SUSY QM involved we need the ground state energy to be zero. Thus we define the shifted harmonic oscillator potential

$$V_1(x) = \frac{1}{2}m\omega x^2 - \frac{\hbar\omega}{2} \ .$$

The eigenfunctions of the Hamiltonian $H_1 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x)$ are the same as those of the original harmonic oscillator (3.27) and its spectrum is only shifted by the constant $\frac{\hbar\omega}{2}$.

Now, let us find the SUSY potential $V_2(x)$. We make use of (3.4)[8] to obtain the superpotential

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'(x)}{\psi_0(x)} = \sqrt{\frac{m}{2}}\omega x \ .$$

[8]Notice that in these relations it makes no difference whether ψ_0 is multiplied by a constant. That's why we don't require ψ_0 to be normalized.

Hence from (3.5)

$$V_2(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x) = \frac{m\omega^2}{2} x^2 + \frac{\hbar\omega}{2} = V_1(x) + \hbar\omega \ . \tag{3.28}$$

One can see that $V_2(x)$ compare to $V_1(x)$ is only shifted by a constant while its "shape" (its x-dependence) remains the same. This may inspire us to make a generalization and find a new quantum mechanical concept – the shape invariance of potentials.

The two SUSY partner potentials $V_1(x)$ and $V_2(x)$ are said to be *shape invariant* if they satisfy the condition

$$V_2(x; a_1) = V_1(x; a_2) + R(a_1) \ , \tag{3.29}$$

where a_1 is a set of parameters, a_2 is a function of a_1 (say $a_2 = f(a_1)$) and the remainder $R(a_1)$ is independent of x.

Consider, for example, the superpotential $W(x) = n \tanh x$, where $n \in \mathbb{N}$. Put $\hbar = 2m = 1$ for simplicity. The two SUSY partner potentials are

$$V_1(x; n) = W^2(x) - W'(x) = n^2 - \frac{n(n+1)}{\cosh^2 x} \ ,$$

$$V_2(x; n) = W^2(x) + W'(x) = n^2 - \frac{(n-1)n}{\cosh^2 x} \ .$$

Obviously

$$V_2(x; n) = V_1(x; n-1) + n^2 - (n-1)^2 \ ,$$

i.e. V_1 and V_2 are shape invariant.

The harmonic oscillator is known to be algebraically solvable by the method of raising and lowering operators.[9] In fact, we shall see right away that if the SUSY is unbroken then arbitrary shape invariant potential (SIP) can be solved (i.e. we can obtain its eigenvalues and eigenfunctions) algebraically.

3.3.2 General Formulas for Bound State Spectrum and Wave Functions of SIPs

Let us start from the SUSY partner Hamiltonians H_1 and H_2 whose eigenvalues and eigenfunctions are related by SUSY which is supposed to be unbroken.

Suppose the potential $V_1(x)$ depends on a set of parameters a_1, i.e.

$$H_1 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x; a_1) \ .$$

In view of the shape invariance condition (3.29), its superpartner H_2 reads

$$H_2 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_2(x; a_1) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x; a_2) + R(a_1) \ .$$

[9] See [3], p.79. Notice that our SUSY operators A and $A^\dagger$ (defined by (3.2)) are nothing else but the lowering (or annihilation) and raising (or creation) operators for the harmonic oscillator.

One can introduce another supersymmetry starting from the shifted Hamiltonian $H_2 - R(a_1) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x; a_2)$, whose ground state energy is zero due to the assumption of unbroken SUSY for the potential $V_1(x; a_2)$. The superpartner of $H_2 - R(a_1)$ is

$$H_3 - R(a_1) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x; a_2) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x; a_3) + R(a_2) \ .$$

One can continue in this manner to construct a series of Hamiltonians H_s, $s = 1, 2, 3, ...$ In each step we suppose that SUSY is unbroken. The series stops when the bound states are exhausted. See Figure 3.6.

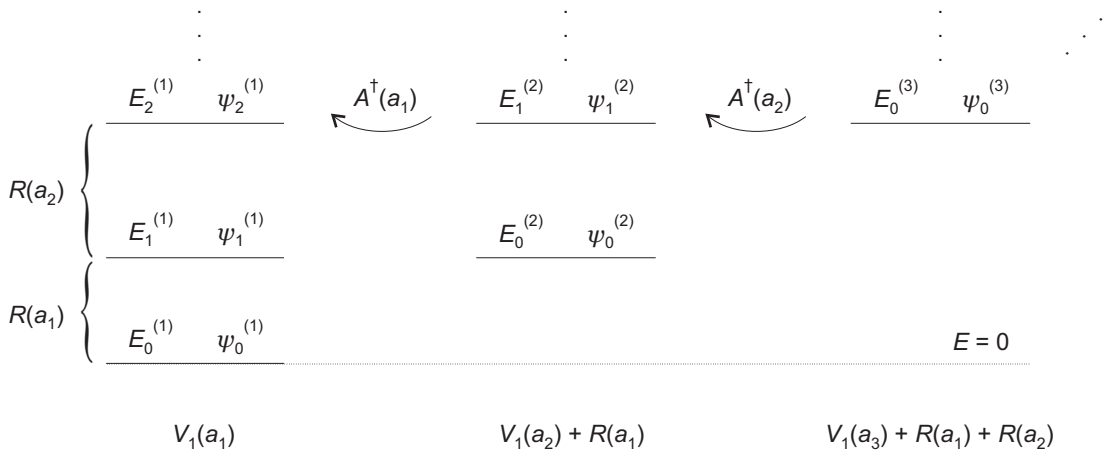

Figure 3.6: A series of SUSY partner potentials connected by the shape invariance condition (3.29). SUSY is supposed to be unbroken in each step.

Generally, for

$$H_s = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x; a_s) + \sum_{k=1}^{s-1} R(a_k)$$

we have the superpartner

$$H_{s+1} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x; a_s) + \sum_{k=1}^{s-1} R(a_k) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x; a_{s+1}) + \sum_{k=1}^{s} R(a_k) \ ,$$

where $a_s = f^{s-1}(a_1)$, i.e. the function f applied $s-1$ times.

It is obvious from the construction (and from the Fig. 3.6) that the n'th energy level of H_1 is coincident with the ground state of the Hamiltonian H_n. Hence the complete eigenvalue spectrum of H_1 is given by

$$E_n^{(1)}(a_1) = \sum_{k=1}^{s} R(a_k) \ , \ E_0^{(1)} = 0 \ .$$

One can use (3.28), which is in fact the shape invariance condition for the harmonic oscillator potential, to easily check that this result is in agreement with the expression for the eigenvalues of a (shifted) harmonic oscillator.

We have seen in the section 3.1.1 that the eigenfunctions of two superpartner Hamiltonians are related by (3.9), i.e.

$$\psi_n^{(1)}(x;a_1) \propto A^\dagger(a_1)\psi_{n-1}^{(2)}(x;a_1) \ .$$

The shape invariance condition (3.29) shows that the potentials $V_2(x;a_1)$ and $V_1(x;a_2)$ have the same eigenfunctions – $\psi_{n-1}^{(2)}(x;a_1) = \psi_{n-1}^{(1)}(x;a_2)$. Thus

$$\psi_n^{(1)}(x;a_1) \propto A^\dagger(a_1)\psi_{n-1}^{(1)}(x;a_2) \ .$$

The n'th state unnormalized eigenfunction $\psi_n^{(1)}(x;a_1)$ for the original Hamiltonian $H_1(x;a_1)$ is therefore given by

$$\psi_n^{(1)}(x;a_1) \propto A^\dagger(a_1)A^\dagger(a_2)...A^\dagger(a_n)\psi_0^{(1)}(x;a_{n+1}) \ .$$

3.3.3 Finding the Shape Invariant Potentials

Now the point is to classify solutions to the shape invariance condition (3.29). Once such a classification is available, then one discovers new SIPs which are solvable by purely algebraic methods. Although the general problem is still unsolved, two classes of solutions have been found.

In the first class, the parameters a_1 and a_2 are related to each other by translation ($a_2 = a_1 + \alpha$). Remarkably enough, all well known analytically solvable potentials discussed in most textbooks on nonrelativistic quantum mechanics (such as Coulomb, Eckart, Rosen-Morse, Pösch-Teller and other potentials) belong to this class.

In the second class, the parameters a_1 and a_2 are related to each other by scaling ($a_2 = qa_1$) and the potentials are only obtained in a series form.

I want to emphasize that the shape invariance is only a sufficient condition for exact solvability, not a necessary condition. There are potentials which have been exactly solved although they are not shape invariant.

One can find more information about shape invariance, including the table of shape invariant potentials, in [1], p.38.

3.4 SUSY Variational Method for Calculating Energy Spectrum and Wave Functions of the Anharmonic Oscillator

The framework of SUSY QM has been very useful in generating several new perturbative methods for calculating the energy spectra and wave functions for one dimensional potentials. They are discussed in more detail in [1]. In this section I briefly show the SUSY-inspired variational method.

Consider, for example, the anharmonic oscillator potential $V(x) = gx^4$ which is not an exactly solvable problem in QM. Let us look for the (normalized) approximate ground state

wave function in the form

$$\psi_0(x;\beta) := \left(\frac{2\beta}{\pi}\right)^{1/4} e^{-\beta x^2} \ , \tag{3.30}$$

where β is the variational parameter.

Minimizing the expression

$$\langle\psi_0(\beta) \mid H \mid \psi_0(\beta)\rangle = \left\langle \psi_0(\beta) \mid -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + gx^4 \mid \psi_0(\beta)\right\rangle = \frac{\beta\hbar}{2m} + \frac{3g}{16\beta^2}$$

with respect to the parameter β yields

$$\beta_0 = \left(\frac{3}{4}\frac{mg}{\hbar}\right)^{1/3} \ ,$$

and the approximate ground state energy

$$E_0 = \langle\psi_0(\beta_0) \mid H \mid \psi_0(\beta_0)\rangle = \left(\frac{3}{4}\right)^{4/3}\left(\frac{\hbar}{m}\right)^{2/3} g^{1/3} \doteq 0.68142\left(\frac{\hbar}{m}\right)^{2/3} g^{1/3}.$$

This is rather good for this crude approximation since the exact ground state energy of the anharmonic oscillator determined numerically is $E_0 = 0.66799\left(\frac{\hbar}{m}\right)^{2/3} g^{1/3}$.

The approximate superpotential W resulting from this variational calculation is (in view of (3.4))

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{1}{\psi_0(x;\beta_0)}\frac{d\psi_0(x;\beta_0)}{dx} = \frac{2\hbar}{\sqrt{2m}}\beta_0 x \ , \tag{3.31}$$

which leads, within this approximation, to the potential

$$V_1(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x) = \frac{2\hbar^2}{m}\beta_0^2 x^2 - \frac{\hbar^2}{m}\beta_0 \ .$$

The (approximate) SUSY partner potential is now

$$V_2(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x) = \frac{2\hbar^2}{m}\beta_0^2 x^2 + \frac{\hbar^2}{m}\beta_0 \ .$$

Since V_2 differs from V_1 by a constant $\frac{2\hbar^2}{m}\beta_0$, the approximate ground state wave function for V_2 is also given by (3.30) with $\beta = \beta_0$. The approximate ground state energy of V_2,[10] is now

$$\langle\psi_0(\beta_0) \mid H_2 \mid \psi_0(\beta_0)\rangle = \langle\psi_0(\beta_0) \mid H_1 \mid \psi_0(\beta_0)\rangle + \frac{2\hbar^2}{m}\beta_0 \ .$$

Thus we find in this harmonic approximation that the energy difference between the ground state and the first excited state of the anharmonic oscillator is

$$\frac{2\hbar^2}{m}\beta_0 = 2\left(\frac{3}{4}\right)^{1/3}\frac{\hbar^2}{m}\left(\frac{mg}{\hbar}\right)^{1/3} \doteq 1.817\frac{\hbar^2}{m}\left(\frac{mg}{\hbar}\right)^{1/3}$$

[10] which is, thanks to supersymmetry, also the first excited state energy of V_1,

which is to be compared with the exact numerical value of $0.863\frac{\hbar^2}{m}\left(\frac{mg}{\hbar}\right)^{1/3}$. This shows that the harmonic approximation breaks down rapidly when we consider the higher energy eigenstates of the anharmonic oscillator.

The approximate (unnormalized) first excited state wave function in this simple approximation is (see Figure 3.1)

$$\psi_1^{(1)} = A^\dagger \psi_0^{(2)} = A^\dagger \psi_0^{(1)} = \left(-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right)\psi_0(x;\beta_0)$$

and substituting from (3.30) and (3.31) gives

$$\psi_1^{(1)}(x) = \left(-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + \frac{2\hbar}{\sqrt{2m}}\beta_0 x\right)\left(\frac{2\beta_0}{\pi}\right)^{1/4} e^{-\beta_0 x^2} = \frac{\hbar}{\sqrt{2m}}\left(\frac{2\beta_0}{\pi}\right)^{1/4} 4\beta_0 x e^{-\beta_0 x^2} \ .$$

To obtain better accuracy it is necessary to extend the number of variational parameters. Calculation using more variational parameters (polynomials times exponentials) gave energy eigenvalues for low lying states accurate to 0.1%.

Chapter 4

Berry Phase

A type of evolution of a physical system which is often of interest in physics is one in which the state of the system returns to its original state after an evolution. We shall call this a cyclic evolution. Now, in quantum mechanics, the initial- and final-state vectors of a cyclic evolution are related by a phase factor $e^{i\varphi_B}$ (so called *Berry phase*), which can have observable consequences.

The Berry phase was firstly published by M.V.Berry in 1984 in the context of adiabatic evolution of a cyclic quantum system. B.Simon explained mathematically the Berry phase as an *anholonomy*[1] in the parameter space. Later on, Y.Aharonov and J.Anandan showed that if the evolution is cyclic in the projective Hilbert space then the anholonomy is not based on the assumption of adiabaticity.

Many further generalizations showed that the Berry phase is a pure artefact of the geometry of the projective Hilbert space and hence it is independent of the actual Hamiltonian (as long as the curve in the projective Hilbert space is the same) and is even independent of whether or not the evolution is cyclic (Pancharatnam's phase).

I aim to present the Berry phase concept without going deeper into geometry of projective Hilbert spaces.

4.1 Phase Change during a Cyclic Quantum Evolution

Let us have a Hilbert space $\mathcal{H}$ of normalized quantum states $|\psi\rangle$. Let us introduce an equivalence relation $\sim$ between two states $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$

$$|\psi_1\rangle \sim |\psi_2\rangle \quad \Leftrightarrow \quad (\exists c \in U(1))(|\psi_1\rangle = c|\psi_2\rangle) \ .^2$$

[1] Anholonomy is a geometrical phenomenon in which nontrivial topology of the configuration manifold causes some variables to fail to return to their original values when others, which drive them, are altered round a cycle. The simplest anholonomy is in the parallel transport of vectors – e.g. the change in the direction of swing of a Foucault pendulum after one rotation of the earth.

[2] $U(1) = \{c \in \mathbb{C} \mid |c| = 1\}$

That is, two states are $\sim$-equivalent if they only differ by a phase factor $e^{i\varphi}$. This equivalence gives rise to the *projective Hilbert space* $\mathcal{P} = \mathcal{H}_{/\sim}$ which consists of equivalence classes

$$[|\psi\rangle] = \{|\psi'\rangle \in \mathcal{H} \mid |\psi'\rangle = c|\psi\rangle, c \in U(1)\}$$

called *rays*. Finally, let Π be the projection map

$$\Pi : \mathcal{H} \to \mathcal{P} \ , \ |\psi\rangle \mapsto [|\psi\rangle] \ .$$

Suppose that the normalized state $|\psi\rangle$ evolves according to the Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle \ , \tag{4.1}$$

such that, for some given T, $|\psi(T)\rangle = e^{i\varphi}|\psi(0)\rangle$, where $\varphi \in \mathbb{R}$. Then $|\psi(t)\rangle$ defines a curve $\mathcal{C} : [0, T] \to \mathcal{H}$ with its projection $\tilde{\mathcal{C}} \equiv \Pi(\mathcal{C})$ being a closed curve in $\mathcal{P}$ (see Fig.4.1).

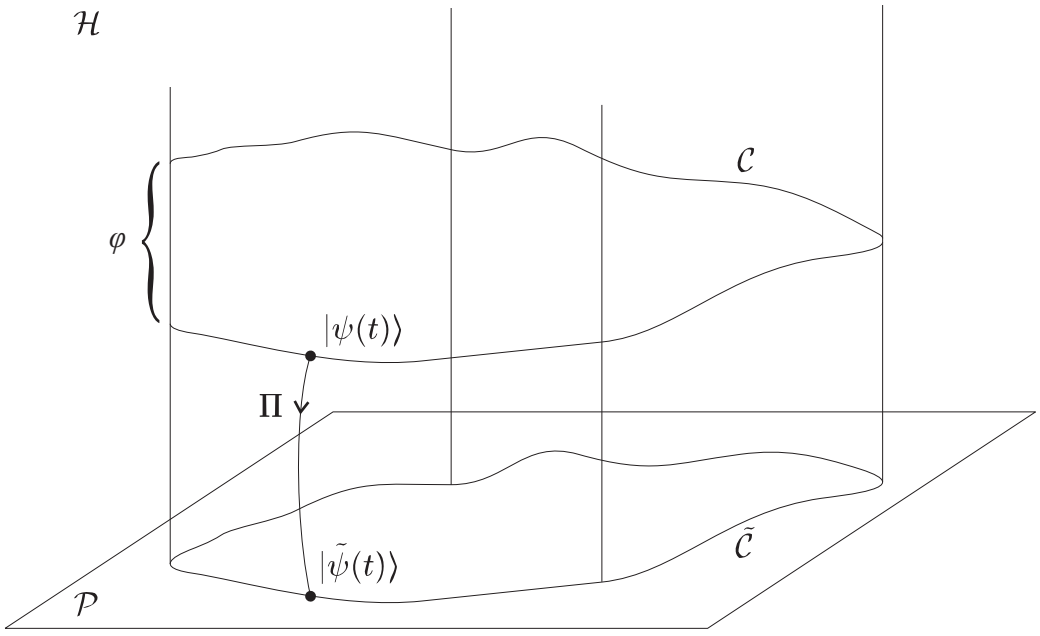

Figure 4.1: A cyclic evolution of a quantum system governed by the Hamiltonian $H(t)$ traces out a curve $\mathcal{C}$. The phase of the wavefunction $|\psi(t)\rangle$ is changed by φ after one cycle. The mapping Π projects the curve $\mathcal{C}$ onto the projective Hilbert space $\mathcal{P}$ (the space of rays) creating a closed curve $\tilde{\mathcal{C}}$.

Now define

$$|\tilde{\psi}(t)\rangle := e^{-if(t)}|\psi(t)\rangle \ , \ f(T) - f(0) := \varphi \ .$$

Then $|\tilde{\psi}(T)\rangle = |\tilde{\psi}(0)\rangle$ and hence $\{|\tilde{\psi}(t)\rangle \mid t \in [0, T]\}$ represents the closed curve $\tilde{\mathcal{C}}$. Inserting $|\psi(t)\rangle = e^{if(t)}|\tilde{\psi}(t)\rangle$ into (4.1) we obtain

$$H(t)|\tilde{\psi}(t)\rangle = i\hbar(i\frac{df}{dt}|\tilde{\psi}(t)\rangle + \frac{d}{dt}|\tilde{\psi}(t)\rangle)$$

and multiplying the equation by (normalized) $\langle\tilde{\psi}(t)|$ yields

$$\langle\psi(t)|H(t)|\psi(t)\rangle = \langle\tilde{\psi}(t)|H(t)|\tilde{\psi}(t)\rangle = -\hbar\frac{df}{dt} + \hbar\langle\tilde{\psi}(t)|i\frac{d}{dt}|\tilde{\psi}(t)\rangle \ .$$

Finally, let's divide the whole equation by $\hbar$ and integrate from 0 to T:

$$\underbrace{\int_0^T \frac{1}{\hbar}\langle\psi(t)|H(t)|\psi(t)\rangle dt}_{=:\,-\varphi_D} = -\underbrace{(f(T)-f(0))}_{=\,\varphi} + \underbrace{\int_0^T \langle\tilde{\psi}(t)|i\frac{d}{dt}|\tilde{\psi}(t)\rangle dt}_{=:\,\varphi_B} \ .$$

I have split the total phase that the state $|\psi(t)\rangle$ acquires during cyclic evolution along the curve $\mathcal{C}$ into two parts:

$$\varphi = \varphi_D + \varphi_B \ .$$

The dynamical part

$$\varphi_D = -\int_0^T \frac{1}{\hbar}\langle\psi(t)|H(t)|\psi(t)\rangle dt$$

depends on the Hamiltonian H, whereas the geometric Berry phase

$$\varphi_B = \int_0^T \langle\tilde{\psi}(t)|i\frac{d}{dt}|\tilde{\psi}(t)\rangle dt \tag{4.2}$$

does not. We should check that this definition of the Berry phase φ_B is correct, i.e. that it does not depend on concrete representation of the curve $\tilde{\mathcal{C}}$, which is a curve in the space of rays $\mathcal{P}$.

Really, if we take, say

$$|\tilde{\tilde{\psi}}(t)\rangle := e^{ig(t)}|\tilde{\psi}(t)\rangle \ ,$$

where $g(0) = g(T)$, then

$$\int_0^T \langle\tilde{\tilde{\psi}}(t)|i\frac{d}{dt}|\tilde{\tilde{\psi}}(t)\rangle dt = \int_0^T \langle e^{ig(t)}\tilde{\psi}(t)|i\frac{d}{dt}|e^{ig(t)}\tilde{\psi}(t)\rangle dt =$$

$$= \int_0^T \Big(-\frac{dg}{dt}\underbrace{\langle\tilde{\psi}(t)|\tilde{\psi}(t)\rangle}_{=\,1} + \langle\tilde{\psi}(t)|i\frac{d}{dt}|\tilde{\psi}(t)\rangle\Big)dt = -\underbrace{(g(T)-g(0))}_{=\,0} + \varphi_B \ .$$

Now, clearly, the same $|\tilde{\psi}(t)\rangle$ can be chosen for every curve $\mathcal{C}'$ for which $\Pi(\mathcal{C}') = \tilde{\mathcal{C}}$ by appropriate choice of $f(t)$, for φ_B is independent of φ. Also, (4.2) can be rewritten as

$$\varphi_B = \int_0^T \langle\tilde{\psi}(t)|i\frac{d}{dt}|\tilde{\psi}(t)\rangle dt = \int_{\tilde{\mathcal{C}}} B \ , \tag{4.3}$$

where

$$B := i\langle\tilde{\psi}|d|\tilde{\psi}\rangle \tag{4.4}$$

is so called *Berry connection 1-form*. Hence φ_B is a geometric property of the unparameterized image of $\tilde{\mathcal{C}}$ in $\mathcal{P}$ only. As a result of its geometric origin, φ_B is observable and measurable (unlike the dynamical phase φ_D).

4.2 Berry Phase in One Dimensional Quantum Mechanics

I remind that it has been discussed in the section 2 that in the case of 1-D Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

with the potential $V(x)$ well defined on the whole real line, the spectrum is non-degenerate and the eigenfunctions can be chosen to be real.

Suppose our Hamiltonian H is a function of a set of parameters R^j. Once we know an eigenstate $\psi_n(x, \mathbf{R})$ of $H(\mathbf{R})$ corresponding to a (non-degenerate) eigenvalue $E_n(\mathbf{R})$, we can calculate the Berry phase which the eigenstate $\psi_n(x, \mathbf{R})$ develops during *adiabatic*[3] evolution along a closed curve Γ in the parameter space.

From (4.3) and (4.4) the Berry phase is

$$\varphi_B = \int_{\tilde{\mathcal{C}}} i\langle\psi_n|d|\psi_n\rangle = \int_\Gamma i\sum_j \langle\psi_n(\mathbf{R})|\frac{\partial}{\partial R^j}|\psi_n(\mathbf{R})\rangle dR^j \ .$$

But if we investigate j-th scalar product in the integral, remembering that $\psi_n(x, \mathbf{R})$ is real valued, we find

$$\langle\psi_n(\mathbf{R})|\frac{\partial}{\partial R^j}|\psi_n(\mathbf{R})\rangle = \int_{\mathbb{R}} \psi_n(x,\mathbf{R})\frac{\partial}{\partial R^j}\psi_n(x,\mathbf{R})dx = \int_{\mathbb{R}} \frac{1}{2}\frac{\partial}{\partial R^j}(\psi_n^2(x,\mathbf{R}))dx =$$

$$= \frac{1}{2}\frac{\partial}{\partial R^j}\int_{\mathbb{R}} \psi_n^2(x,\mathbf{R})dx \ .^4$$

Hence φ_B is an integral from an exact 1-form along a closed loop and therefore the Berry phase is

$$\varphi_B = 0$$

and no anholonomy occurs.

It seems that if we want to observe some non-trivial geometric phase, we need to consider more general class of potentials. In order to violate the assumptions about the shape of the potential $V(x)$ stated in the section 2, which led to the Berry phase equal to zero, let us admit that the potential may have some singular point, say $x = 0$.

This singularity divides the real line $\mathbb{R}$ into two half-lines $\mathbb{R}_+$ and $\mathbb{R}_-$, where the time-independent Schrödinger equation is to be solved separately. Once we know the eigenfunctions $|\psi_n^{(+)}\rangle$ and $|\psi_n^{(-)}\rangle$ defined on $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively, the question is how to connect them at the singular point to obtain an eigenfunction for the whole system. This connection can be described by some boundary conditions at $x = 0$, depending eventually on a set of parameters. These boundary conditions may be such that the functions $|\psi_n^{(\pm)}\rangle$ (corresponding to the same

[3] Adiabatic means that the change of parameters $\mathbf{R}$ is slow enough so that if the system starts as the n'th eigenstate of $H(\mathbf{R}_0)$ for some parameter configuration $\mathbf{R}_0 \in \Gamma$, it remains the n'th eigenstate of $H(\mathbf{R})$ for all $\mathbf{R} \in \Gamma$.

[4] Of course, if it is legal to commute $\int_{\mathbb{R}}$ and $\frac{\partial}{\partial R^j}$.

energy level) cannot be both real valued. Thus we can construct a closed loop in the parameter space and calculate the Berry phase which may be possibly non-zero.

In [9], sec. 3.2 the Berry phase is calculated for a 1-D free particle in a box with removed origin. The boundary conditions at the origin are discussed as well. They are, in fact, the main topic of the article. The Berry phase indeed turns out to be non-zero.

Chapter 5

Conclusion

We have got acquainted with the fundamental ideas of SUSY QM. We have seen how the concept of SUSY partner Hamiltonians can be useful when one needs to calculate reflection and transmission coefficients or eigenvalues and eigenfunctions of certain potentials, which would otherwise be very difficult (or laborious) to solve. For more information I must once again recommend the reference [1], where the shape invariance is discussed in much more detail, where various perturbation methods are shown and where there are some applications of SUSY QM that I didn't even mention in this thesis (e.g. SUSY and the Dirac equation, path integrals and SUSY etc.).

The last chapter was dedicated to the Berry phase. The aim of the further research is to somehow relate the two topics – SUSY QM and the Berry phase. One may suspect that there could be some relation between the Berry phases corresponding to the eigenfunctions of two SUSY partner Hamiltonians which depend on some cyclically altered parameters. This suspicion is, of course, motivated by the fact that the eigenfunctions of the SUSY partners are related in the known way via the operators A and $A^\dagger$ as discussed in the section 3.1.1. However, to obtain some nontrivial results one will probably have to start to deal with more general potentials (recall the arguments in section 4.2).

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