Green Function of the Double-Fractional Fokker-Planck Equation

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Motivation

Generating functional of a QFT:

$$\mathcal{N} \int \mathcal{D}\phi(x) e^{\frac{i}{\hbar}\left\{\mathcal{A}[\phi]+\int d^4x\, j(x)\phi(x)\right\}} = e^{\frac{i}{\hbar}\left\{\Gamma[\Phi]+\int d^4x\, j(x)\Phi(x)\right\}}$$

where $\Phi(x) \equiv \langle \phi(x) \rangle$, and $\Gamma[\Phi]$ is the effective action.

For ϕ^4-theory at strong coupling (e.g., Bose-Einstein condensates at Feshbach resonance), effective action develops anomalous powers.

Extremization of $\Gamma[\Phi] \to$ fractional Gross-Pitaevskii equation

$$\left[(\partial_t)^{1-\gamma} + D_\lambda (-\Delta_{\mathbf{x}})^{\lambda/2} + \frac{\delta+1}{4} g_c |\Phi(\mathbf{x},t)|^{\delta-1} \right] \Phi(\mathbf{x},t) = 0$$

H. Kleinert, EPL **100**, 10001 (2012)
H. Kleinert, J. Phys. B **46**, 175401 (2013)

Fokker-Planck equation

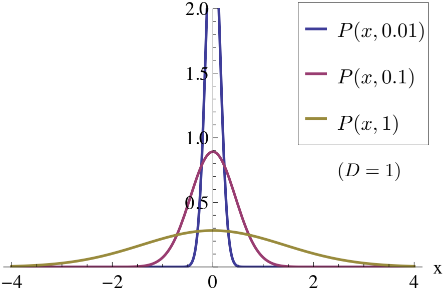
Fokker-Planck (or diffusion, or heat) equation:

$$\left[\partial_t + \hat{H}\right] P(x,t) = 0 \quad , \quad P(x,0) = \delta(x)$$

where

$$\hat{H} = D\hat{p}^2 \quad , \quad \hat{p} \equiv -i\partial_x$$

$$\begin{aligned} P(x,t) &= e^{-t\hat{H}}\delta(x) \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-tDp^2} e^{-ipx} \\ &= \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi Dt}} \quad \ldots \ \textit{Gaussian} \end{aligned}$$



$$\left[\partial_t + \hat{H}\right]\left[\theta(t)P(x,t)\right] = \delta(x)\delta(t)$$

Single-fractional Fokker-Planck equation

Generalized Hamiltonian:

$$\hat{H} = D_\lambda(\hat{p}^2)^{\lambda/2}$$

Lévy stable distribution:

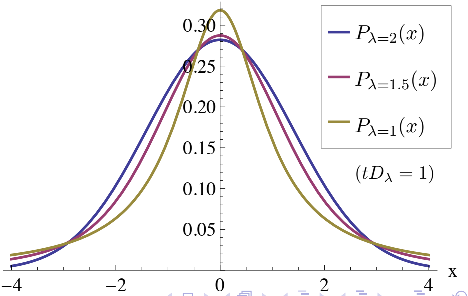
$$P(x,t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-tD_\lambda |p|^\lambda} e^{-ipx}$$

$\lambda = 2$: Gaussian

$\lambda = 1$: Cauchy-Lorentz

$$P(x,t) = \frac{1}{\pi} \frac{D_1 t}{(D_1 t)^2 + x^2}$$

Heavy tails (power-law decay)



Lévy stable distributions

Stability:

$$X_1 \sim Levy \quad , \quad X_2 \sim Levy \quad \Rightarrow \quad X_1 + X_2 \sim Levy$$

Characteristic function:

$$\exp\left[ip\mu - |cp|^\lambda \left(1 - i\beta sgn(p)\Phi\right)\right] \quad , \quad \Phi = \begin{cases} \tan(\lambda\pi/2) & \lambda \neq 1 \\ -(2/\pi)\log|p| & \lambda = 1 \end{cases}$$

λ ...tail power ($\sim |x|^{-1-\lambda}$), c ...width, μ ...shift of origin, β ...asymmetry

Generalized central limit theorem:

$$i.i.d. \quad X_1, \ldots, X_n \quad \sim \quad \text{“}\textit{distribution with possibly infinite variance}\text{''}$$

$$\lim_{n\to\infty} \frac{X_1 + \ldots + X_n}{\mathcal{N}_n} \quad \sim \quad Levy$$

Double-fractional Fokker-Planck equation

Double-fractional Fokker-Planck equation:

$$\left[(\partial_t)^{1-\gamma} + D_\lambda(-\Delta_x)^{\lambda/2}\right] P(x,t) = \delta(x)\delta(t)$$

(for $\gamma = -1 \ \rightarrow$ wave equation)

Fractional derivative: $(\partial_t)^{1-\gamma} e^{iEt} = (iE)^{1-\gamma} e^{iEt} \rightarrow$ Integral operator:

$$(\partial_t)^{1-\gamma} f(t) = (\partial_t)^{1-\gamma} \int_{-\infty}^{\infty} dt' \delta(t-t') f(t') = \int_{-\infty}^{t} dt' \frac{(t-t')^{\gamma-2}}{\Gamma(\gamma-1)} f(t')$$

Relativistic Hamiltonian ($\hat{E} = -i\partial_t$):

$$\hat{\mathcal{H}} = (i\hat{E})^{1-\gamma} + D_\lambda(\hat{p}^2)^{\lambda/2} \quad \rightarrow \quad \hat{\mathcal{H}} P(x,t) = \delta(x)\delta(t)$$

$$\Rightarrow P(x,t) = \frac{1}{\hat{\mathcal{H}}}\delta(x)\delta(t) = \int_0^{\infty} ds\ e^{-s\hat{\mathcal{H}}}\delta(x)\delta(t)$$

(using "Schwinger trick" representation)

Smeared-time representation of the Green function

$$P(x,t) = \int_0^\infty ds\ P_X(x,s) P_T(t,s)$$

where

$$P_X(x,s) = e^{-sD_\lambda (\hat{p}^2)^{\lambda/2}} \delta(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-sD_\lambda |p|^\lambda} e^{-ipx}$$

$$P_T(t,s) = e^{-s\hat{E}^{1-\gamma}} \delta(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-s(-iE)^{1-\gamma}} e^{-iEt}$$

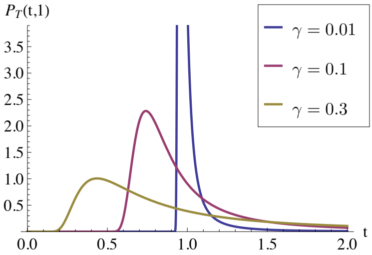
$$\left[P_T(t,s)|_{\gamma=0} = \delta(t-s) \quad \Rightarrow \quad P(x,t) = \theta(t) P_X(x,t) \right]$$

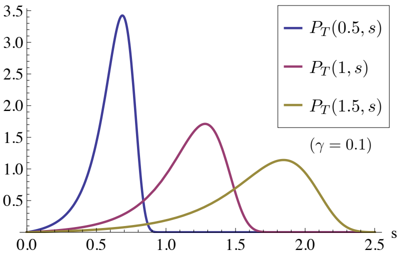
$t \ldots$ physical time
$s \ldots$ pseudotime

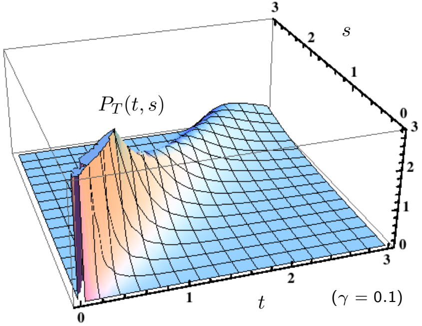
Normalization:

$$\int_{-\infty}^{\infty} dx\, P(x,t) = \int_0^\infty ds\ P_T(t,s) = \frac{\theta(t) t^{-\gamma}}{\Gamma(1-\gamma)}$$

Time-smearing distribution P_T







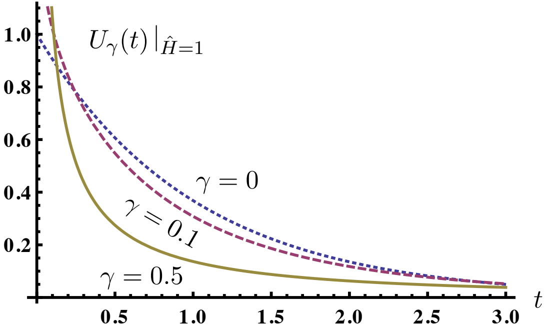
Deformed evolution operator

$$P(x,t) = \hat{U}_\gamma(t)\delta(x)$$

where

$$\hat{U}_\gamma(t) = \theta(t)t^{-\gamma}E_{1-\gamma,1-\gamma}(-t^{1-\gamma}\hat{H}) \quad \xrightarrow{\gamma=0} \quad \theta(t)\, e^{-t\hat{H}}$$

and $E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}$ is the Mittag-Leffler function

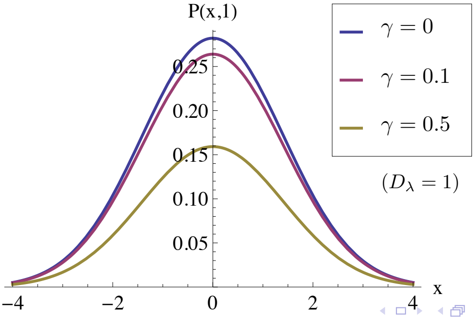


Fox H function representation of the Green function

$$P(x,t) = \frac{t^{-\gamma}}{\sqrt{\pi}|x|} H^{2,1}_{2,3}\left(\frac{|x|^\lambda}{\ell_t^\lambda} \,\middle|\, \begin{matrix} (1,1),(1-\gamma,1-\gamma) \\ (1,1),(1/2,\lambda/2),(1,\lambda/2) \end{matrix} \right)$$

where $\ell_t = 2(D_\lambda t^{1-\gamma})^{1/\lambda}$, and

$$H^{2,1}_{2,3}(\dots) = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{\Gamma(1+z)\Gamma(\frac{1}{2}+\frac{\lambda}{2}z)\Gamma(-z)}{\Gamma(-\frac{\lambda}{2}z)\Gamma(1-\gamma+(1-\gamma)z)} \left[\frac{|x|^\lambda}{\ell_t^\lambda}\right]^{-z}$$



Path integral

$$(x_b\, t_b\, s \,|\, x_a\, t_a\, 0) := \int\limits_{x(0)=x_a,t(0)=t_a}^{x(s)=x_b,t(s)=t_b} \mathcal{D}x\mathcal{D}t\mathcal{D}p\mathcal{D}E\; e^{\int_0^s ds' \left[i(p\frac{dx}{ds'}+E\frac{dt}{ds'})-\mathcal{H}(p,E)-V(x,t)\right]}$$

$(V(x,t)\ldots$ external potential)

$$P(x,t) = \int_0^\infty ds\, (x\, t\, s \,|\, 0\, 0\, 0)|_{V(x,t)=0}$$

For $\mathcal{H}(p,E) = iE + H(p)$, $\int \mathcal{D}E \exp\left[\int_0^s ds'\, iE(\frac{dt}{ds'}-1)\right] = \prod_{s'=0}^s \delta(\frac{dt}{ds'}-1)$
$\rightarrow$ Traditional phase-space path integral

$$\int_0^\infty ds\, (x_b\, t_b\, s \,|\, x_a\, t_a\, 0) = \int\limits_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x\mathcal{D}p\; e^{\int_{t_a}^{t_b} dt \left[ip\frac{dx}{dt}-H(p)-V(x,t)\right]}$$

Thank you for your attention.

H. Kleinert and V. Zatloukal, *Green function of the double-fractional Fokker-Planck equation: Path integral and stochastic differential equations*, Phys. Rev. E **88**, 052106 (2013) [arXiv:1503.01667]