Classical field theories from Hamiltonian constraint:

Canonical equations of motion and local Hamilton-Jacobi theory

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Motivation

Consider a non-relativistic mechanical system with Hamiltonian $H_0(\mathbf{x}, \mathbf{p})$:

Canonical equations of motion:

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H_0}{\partial \mathbf{p}} \quad , \quad \frac{d\mathbf{p}}{dt} = -\frac{\partial H_0}{\partial \mathbf{x}} \tag{1}$$

Hamilton-Jacobi equation: $S(\mathbf{x}, t)$

$$\frac{\partial S}{\partial t} + H_0(\mathbf{x}, \frac{\partial S}{\partial \mathbf{x}}) = 0 \tag{2}$$

Quantization & Schrödinger equation: $\mathbf{p} \to -i\hbar\,\partial/\partial\mathbf{x}$

$$\left[-i\hbar\frac{\partial}{\partial t} + H_0(\mathbf{x}, -i\hbar\frac{\partial}{\partial \mathbf{x}})\right]\psi(\mathbf{x}, t) = 0 \tag{3}$$

Motivation

Our goal: **Hamiltonian formulation of field theory**

Today: Classical field theory
(generalized: momentum, canonical equations, Hamilton-Jacobi)
[V. Zatloukal, *Classical field theories from Hamiltonian constraint: Canonical equations of motion and local Hamilton-Jacobi theory*, arXiv:1504.08344 (2015)]

Someday: Quantization
(generalized: momentum operator, wavefunctions, Schrödinger equation)
See the proposal [I. V. Kanatchikov, arXiv:1312.4518 (2013)]

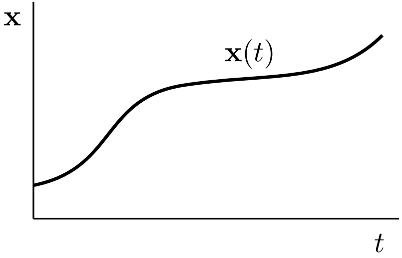
Outline

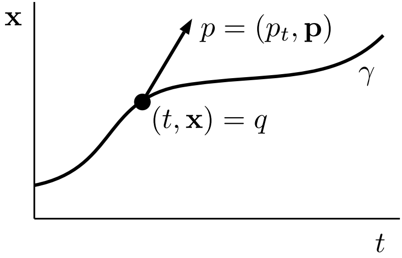
- Partial observables and Relativistic configuration space
- Variational principle with Hamiltonian constraint
- Canonical equations of motion
- Local Hamilton-Jacobi theory
- Examples:
  - Non-relativistic Hamiltonian mechanics
  - Scalar field theory
  - String theory

Partial observables and Relativistic configuration space

Non-relativistic mechanics:
Hamiltonian $H_0(\mathbf{x}, \mathbf{p})$
Trajectories are functions $\mathbf{x}(t)$

Relativistic formalism:
Curves $\gamma = \{q = (t, \mathbf{x}) \mid f(t, \mathbf{x}) = 0)\}$
Hamiltonian constraint
$H(q, p) = p_t + H_0(\mathbf{x}, \mathbf{p}) = 0$

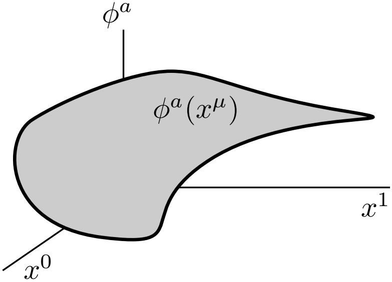


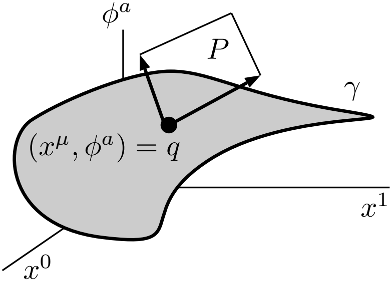


Relativistic formalism is more compact, symmetric, and allows to describe both non-relativistic and relativistic mechanical systems (e.g., free relativistic particle: $H = p_\mu p^\mu - m^2$).

Partial observables and Relativistic configuration space

Field theory: functions $\phi^a(x^\mu) \to$ surfaces $\gamma = \{q = (x^\mu, \phi^a) \,|\, f(x, \phi) = 0\}$





Following [C. Rovelli, *Quantum Gravity*, Cambridge Univ. Press (2004), Ch. 3]

$t, \mathbf{x}, \phi$ ... *partial observables*
$\mathcal{C} = \{q\}$ ... *configuration space* – $N + D$-dimensional, Euclidean
$\gamma \subset \mathcal{C}$ ... *motions* – D-dim., correlations among partial observables

Mathematical formalism

We use the mathematical formalism of **geometric algebra and calculus**:
[D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, (1987)]
See also [C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, (2007)]

$A \cdot B$ ... inner product

$A \wedge B$ ... outer product

$\partial_q \equiv \sum_{j=1}^{N+D} e_j \, e_j \cdot \partial_q$ ... vector derivative (with respect to point in $\mathcal{C}$)

Variational principle with Hamiltonian constraint

$d\Gamma$ ... oriented surface element of γ
P ... multivector of grade D

Variational principle

A surface $\gamma_{\rm cl}$ with boundary $\partial\gamma_{\rm cl}$ is a physical motion, if the couple $(\gamma_{\rm cl}, P_{\rm cl})$ extremizes the (action) functional

$$\mathcal{A}[\gamma, P] = \int_\gamma P(q) \cdot d\Gamma(q) \tag{4}$$

in the class of pairs (γ, P), for which $\partial\gamma = \partial\gamma_{\rm cl}$, and P defined along γ satisfies the **Hamiltonian constraint**

$$H(q, P(q)) = 0 \qquad \forall q \in \gamma. \tag{5}$$

cf. Ch. 3.3.2 in [C. Rovelli, *Quantum Gravity*, Cambridge Univ. Press (2004)]

Variational principle with Hamiltonian constraint

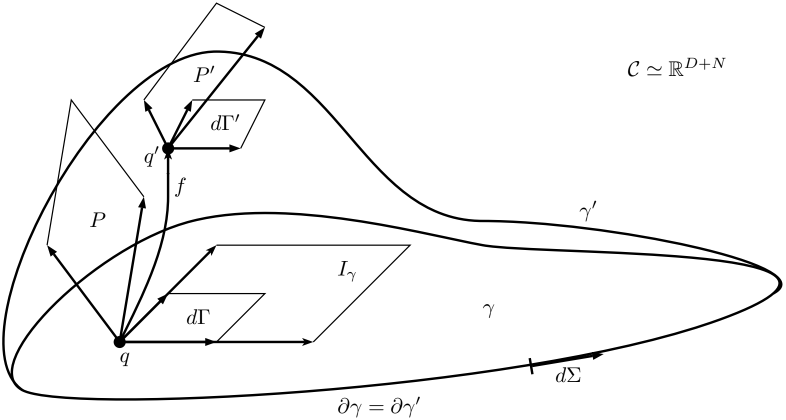


Figure: Variational principle.

Canonical equations of motion

$$\mathcal{A}[\gamma, P, \lambda] = \int_\gamma [P(q) \cdot d\Gamma(q) - \lambda(q) H(q, P(q))] \tag{6}$$

Lagrange multiplier $\lambda(q)$ – infinitesimal ($\lambda \sim |d\Gamma|$)

Variation with respect to γ, P, λ yields:

(see [V. Zatloukal, arXiv:1504.08344 (2015)] for detailed derivation)

Canonical equations of motion

Canonical equations of motion

Physical motions $\gamma_{\rm cl}$ *are obtained by solving the system of equations*

$$\lambda\,\partial_P H(q,P) = d\Gamma, \tag{7a}$$

$$(-1)^D \lambda\,\dot{\partial}_q H(\dot{q},P) = \begin{cases} d\Gamma\cdot\partial_q P & \text{for } D=1 \\ (d\Gamma\cdot\partial_q)\cdot P & \text{for } D>1, \end{cases} \tag{7b}$$

$$H(q,P) = 0. \tag{7c}$$

(7a) "Velocity–momentum" relation
(7b) "Force = Change in momentum"
(7c) Hamiltonian constraint

Local Hamilton-Jacobi theory

Suppose $P(q) = \partial_q \wedge S(q)$ on an open subset of $\mathcal{C}$, for a $(D-1)$-vector S

IF (see Eq. (7c))

Local Hamilton-Jacobi equation

$$H(q, \partial_q \wedge S) = 0, \tag{8}$$

AND (see Eq. (7a))

$$\lambda\, \partial_P H(q, \partial_q \wedge S) = d\Gamma, \tag{9}$$

THEN

the second canonical equation (7b) is fulfilled automatically.

Local Hamilton-Jacobi theory

If we find a family of solutions $S(q;\alpha)$, where α is a continuous parameter, by differentiation ∂_α we obtain:

$D = 1$: Constant of motion

$$d\Gamma \cdot \partial_q(\partial_\alpha S) = 0 \quad \Rightarrow \quad \partial_\alpha S(q;\alpha) = \beta \qquad \forall q \in \gamma_{\mathrm{cl}}, \tag{10}$$

With N independent parameters $\alpha_1, \ldots, \alpha_N$, we determine γ_{cl} from implicit equations (10). (Note: $\mathcal{C} \simeq \mathbb{R}^{N+1}$)

$D > 1$: Continuity equation

$$(d\Gamma \cdot \partial_q) \cdot (\partial_\alpha S) = 0 \quad \Rightarrow \quad \int_{\bar{\gamma}_{\mathrm{cl}}} (d\Gamma \cdot \partial_q) \cdot (\partial_\alpha S) = \int_{\partial\bar{\gamma}_{\mathrm{cl}}} d\Sigma \cdot (\partial_\alpha S) = 0 \tag{11}$$

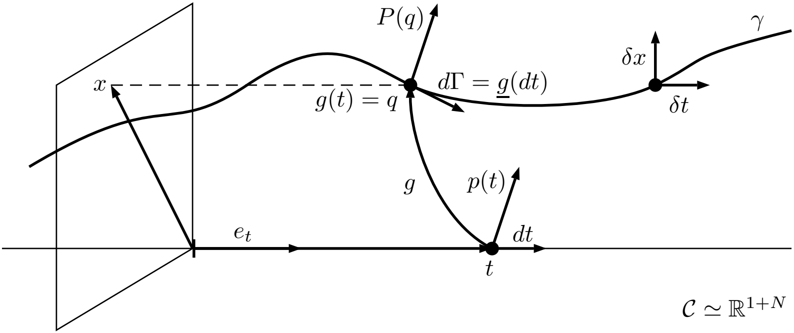
where $\bar{\gamma}_{\mathrm{cl}}$ is in general a subset of γ_{cl}.

Example 1: Non-relativistic Hamiltonian mechanics

Consider $D = 1$, and take

$$H(q, P) = P \cdot e_t + H_0(q, P), \quad (12)$$

where $e_t \cdot \partial_P H_0 = 0$, H_0 ... non-relativistic Hamiltonian.



$$\gamma_{\mathrm{cl}} = \{q = g(t) = t + x(t) \,|\, t \in \mathrm{span}\{e_t\} \simeq \mathbb{R}\} \quad (13)$$

Example 1: Non-relativistic Hamiltonian mechanics

Denote $p(t) \equiv P(g(t))$. Canonical equations (7) reduce to

Hamilton's canonical equations:

$$e_t \cdot \partial_t x = \partial_p H_0(q, p) \quad , \quad e_t \cdot \partial_t \, e_x \cdot p = -e_x \cdot \partial_q H_0(q, p) \tag{14}$$

and Energy conservation law:

$$e_t \cdot \partial_t H_0(q(t), p(t)) = e_t \cdot \partial_q H_0(q, p(t))|_{q=g(t)} = 0 \tag{15}$$

(assuming H_0 does not depend on time t explicitly.)

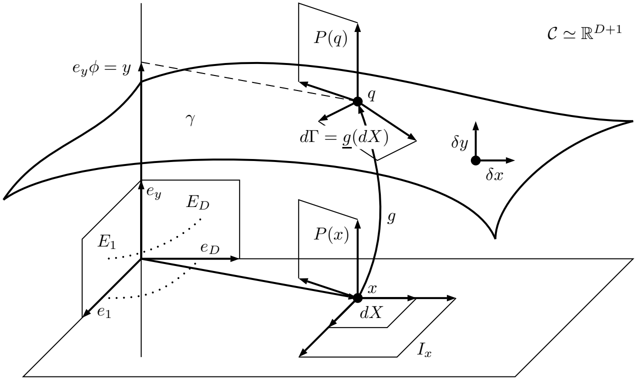
Hamilton-Jacobi equation: ($S(q)$ is scalar function)

$$H(q, \partial_q S) = e_t \cdot \partial_q S + H_0(q, \partial_q S) = 0 \tag{16}$$

Consider $D > 1, N = 1$, and take

$$H(q,P) = P \cdot I_x + H_{DW}(q,P), \tag{17}$$

where $I_x \cdot \partial_P H_{DW} = 0$, H_{DW} ... De Donder-Weyl Hamiltonian.



$$\gamma_{\rm cl} = \{q = g(x) = x + y(x) \,|\, x \in \mathrm{span}\{e_1, \ldots, e_D\} \simeq \mathbb{R}^D\}. \tag{18}$$

Example 2: Scalar field theory

Denote $P(x) \equiv P(g(x))$, $E_j \equiv I_x e_j e_y$, $\widetilde{E}_j$ ... reversion of E_j.
Canonical equations (7) reduce to

De Donder-Weyl equations:

$$e_j \cdot \partial_x \, e_y \cdot y = \widetilde{E}_j \cdot \partial_P H_{DW} \quad , \quad e_j \cdot \partial_x E_j \cdot P = -e_y \cdot \partial_q H_{DW} \tag{19}$$

and Continuity equation for the energy-momentum tensor:

$$e_k \cdot \partial_x \mathcal{T}_{jk} = 0 \tag{20}$$

(assuming H_{DW} does not depend on x explicitly.)

In particular, for $H_{DW} = \frac{1}{2}\sum_{j=1}^{D}(P \cdot E_j)^2 + V(\phi)$, Eqs. (19) simplify,

$$\partial_x^2 \phi = -\partial_\phi V(\phi) \quad , \quad \phi \equiv e_y \cdot y, \tag{21}$$

and $\mathcal{T}_{jk} = -\delta_{jk}\mathcal{L}(\phi, \partial_x \phi) + (e_j \cdot \partial_x \phi)(e_k \cdot \partial_x \phi)$, where $\mathcal{L} = \frac{1}{2}(\partial_x \phi)^2 - V(\phi)$.

Example 2: Scalar field theory

Hamilton-Jacobi equation:

$$I_x \cdot (\partial_q \wedge S) + H_{DW}(q, \partial_q \wedge S) = 0 \tag{22}$$

In particular, for H_{DW} specified above, and a vector $s(q) \equiv S(q) \cdot I_x$,

$$\partial_q \cdot s + \frac{1}{2}(e_y \cdot \partial_q\, s)^2 + V(\phi) = 0. \tag{23}$$

Coincides with Hamilton-Jacobi equation of Weyl.
(See e.g. [H. Kastrup, Phys. Rep. **101** (1983), 1-167])

Example 3: String theory

$\mathcal{C}$ = target space (Euclidean): dim $= N + D$
γ = world-sheet: dim $= D$

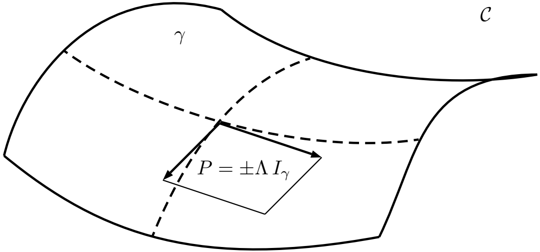
Consider the Hamiltonian

$$H(P) = \frac{1}{2}(|P|^2 - \Lambda^2),$$

where $|P|^2 \equiv \widetilde{P} \cdot P$.

Canonical Eqs. (7) imply:

$$d\Gamma = \lambda \widetilde{P} \quad , \quad |d\Gamma| = |\lambda| \Lambda$$



$I_\gamma \equiv d\Gamma/|d\Gamma| = \pm P/\Lambda$ ... unit pseudoscalar of γ

Example 3: String theory

Nambu-Goto action:

$$\int_\gamma P \cdot d\Gamma = \int_\gamma \frac{1}{\lambda}|d\Gamma|^2 = \pm \Lambda \int_\gamma |d\Gamma|, \tag{24}$$

$\rightarrow \gamma_{\mathrm{cl}}$ is a *minimal surface* (mean curvature vanishes)

$D = 1$: Relativistic particle

$$I_\gamma \cdot \partial_q I_\gamma = 0 \tag{25}$$

$D > 1$: String or membrane

$$(I_\gamma \cdot \partial_q) \cdot I_\gamma = 0 \tag{26}$$

Hamilton-Jacobi equation:

$$|\partial_q \wedge S| = \Lambda \tag{27}$$

Example 3: Relativistic particle – physical motions

Integrating ($|d\Gamma|$-multiple of) Eq. (25) along γ from q_0 to q, and applying the *Fundamental theorem of geometric calculus*,

$$0 = \int_{q_0}^{q} d\Gamma \cdot \partial_q I_\gamma = I_\gamma(q) - I_\gamma(q_0) \tag{28}$$

$\Rightarrow$ I_γ is constant along a physical motion
$\Rightarrow$ γ_{cl} are straight lines in $\mathcal{C}$:

$$\gamma_{\mathrm{cl}} = \{q = v\tau + q_0 \,|\, \tau \in \mathbb{R}\} \tag{29}$$

($q_0 \in \mathcal{C}$ and v is arbitrary constant vector.)

Summary

- We showed how field theory can be formulated using Hamiltonian constraint between partial observables and generalized momentum: $\mathcal{A} = \int_\gamma P \cdot d\Gamma \ , \ H(q, P) = 0$
- We derived canonical equations of motion: $\lambda \, \partial_P H(q, P) = d\Gamma \ , \ (-1)^D \lambda \, \dot{\partial}_q H(\dot{q}, P) = (d\Gamma \cdot \partial_q) \cdot P$
- and Hamilton-Jacobi equation: $H(q, \partial_q \wedge S) = 0$
- Scalar field theory and string theory formulated in a common framework.

Thank you for your attention.